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A one-dimensional analysis of real and complex turbulence and the Maxwell set for the stochastic Burgers equation*

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Abstract

The inviscid limit of the Burgers equation, with body forces white noise in time, is discussed in terms of the level surfaces of the minimizing Hamilton–Jacobi function and the classical mechanical caustic and their algebraic pre-images under the classical mechanical flow map. The problem is analysed in terms of a reduced (one-dimensional) action function using a circle of ideas due to Arnol'd, Cayley and Klein. We characterize those parts of the caustic which are singular, and give an explicit expression for the cusp density on caustics and level surfaces. By considering the double points of level surfaces we find an explicit formula for the Maxwell set in the two-dimensional polynomial case, and we extend this to higher dimensions using a double discriminant of the reduced action, solving a long-standing problem for Hamiltonian dynamical systems. When the pre-level surface touches the pre-caustic, the geometry (number of cusps) on the level surface changes infinitely rapidly causing 'real turbulence'. Using an idea of Klein, it is shown that the geometry (number of swallowtails) on the caustic also changes infinitely rapidly when the real part of the pre-caustic touches its complex counterpart, causing 'complex turbulence'. These are both inherently stochastic in nature, and we determine their intermittence in terms of the recurrent behaviour of two processes.

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(Some figures in this article are in colour only in the electronic version)

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1. Introduction

Burgers equations have been used in studying turbulence and in modelling the large-scale structure of the universe [1], as well as to obtain detailed asymptotics for stochastic Schrödinger and heat equations [2]. In the deterministic case, they have also played a part in Arnol'd's pioneering work on caustics and Maslov's seminal works in semiclassical quantum mechanics which inspired much of the early work in this subject [3].

We consider the stochastic, viscous Burgers equation for the velocity field $v^\mu(x,t) \in \mathbb{R}^d$, where $x \in \mathbb{R}^d$, $t > 0$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla)v^\mu = \frac{\mu^2}{2}\Delta v^\mu - \nabla V(x) - \epsilon \nabla k_t(x)\dot{W}_t,$$
$$v^\mu(x,0) = \nabla S_0(x) + \mathrm{O}(\mu^2),$$

$\dot{W}_t$ being white noise and μ^2 the coefficient of viscosity being small.

We are interested in the advent of discontinuities in

$$v^0(x,t) = \lim_{\mu \searrow 0} v^\mu(x,t).$$

The corresponding *Stratonovich heat equation* for the scalar temperature $u^\mu(x,t) \in \mathbb{R}$, reads

$$\frac{\partial u^\mu}{\partial t} = \frac{\mu^2}{2}\Delta u^\mu + \mu^{-2}V(x)u^\mu + \epsilon\mu^{-2}k_t(x)u^\mu \circ \dot{W}_t,$$
$$u^\mu(x,0) = \exp\left(-\frac{S_0(x)}{\mu^2}\right)T_0(x),$$

where the convergence factor T_0 is related to the initial Burgers fluid density. Here the connection is the *Hopf–Cole transformation*

$$v^\mu(x,t) = -\mu^2\nabla \ln u^\mu(x,t).$$

Following Donsker, and Freidlin and Wentzell [4], as $\mu \to 0$,

$$-\mu^2 \ln u^\mu(x,t) \to \inf_{X(0)}[S_0(X(0)) + A(X(0),x,t)] = \mathcal{S}(x,t),$$

where

$$A(X(0),x,t) = \inf_{\substack{X(s)\\ X(t)=x}} A[X],$$
$$A[X] = \frac{1}{2}\int_0^t \dot{X}^2(s)\,\mathrm{d}s - \int_0^t V(X(s))\,\mathrm{d}s - \epsilon\int_0^t k_s(X(s))\,\mathrm{d}W_s.$$

This gives the *minimal entropy solution* of the Burgers equation [5]. Necessary conditions for X to be an extremizer of $\mathcal{A}[X] = A[X] + S_0(X(0))$ are

$$\mathrm{d}\dot{X}(s) + \nabla V(X(s))\,\mathrm{d}s + \epsilon\nabla k_s(X(s))\,\mathrm{d}W_s = 0, \qquad \dot{X}(0) = \nabla S_0(X(0)).$$

Minimizing $\mathcal{A}[X]$ over $X(0)$ gives $\mathcal{S}(x,t)$, the minimal solution of the *Hamilton–Jacobi equation*

$$\mathrm{d}S_t + \left(\tfrac{1}{2}|\nabla S_t|^2 + V(x)\right)\mathrm{d}t + \epsilon k_t(x)\,\mathrm{d}W_t = 0, \qquad S_{t=0}(x) = S_0(x).$$

Definition 1.1. *The stochastic wavefront at time t is defined to be the set*

$$\mathcal{W}_t = \{x : \mathcal{S}(x,t) = 0\}.$$

For small μ, the heat equation solution $u^\mu(x,t)$ switches *continuously* from being exponentially large to small as we cross the wavefront $\mathcal{W}_t$. However, u^μ and v^μ can also switch *discontinuously* as we now explain.

Define the classical flow map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\mathrm{d}\dot{\Phi}_s + \nabla V(\Phi_s)\,\mathrm{d}s + \epsilon \nabla k_s(\Phi_s)\,\mathrm{d}W_s = 0, \qquad \Phi_0 = \mathrm{id}, \quad \dot{\Phi}_0 = \nabla S_0.$$

Since $X(t) = x$ by definition,

$$X(s) = \Phi_s \Phi_t^{-1} x,$$

where we accept that $x_0(x,t) = \Phi_t^{-1} x$ is not necessarily unique. Given some regularity and boundedness, the global inverse function theorem gives a caustic time $T(\omega) > 0$ such that for $0 < s < T(\omega)$, Φ_s is a random diffeomorphism. For $t < T(\omega)$,

$$v^0(x,t) = \dot{\Phi}_t\big(\Phi_t^{-1} x\big)$$

is a classical solution of the Burgers equation with probability 1.

The method of characteristics suggests that discontinuities in $v^0(x,t)$ are associated with non-uniqueness of real $x_0(x,t)$. This is related to when an infinitesimal volume of points $\mathrm{d}x_0$ focus into zero volume $\mathrm{d}X(t)$ under the classical flow map Φ_t.

Definition 1.2. *The caustic C_t is the set of points x found by eliminating x_0 between*

$$\det\left(\frac{\partial X(t)}{\partial x_0}\right) = 0 \qquad \textit{and} \qquad x = \Phi_t(x_0).$$

The pre-caustic $\Phi_t^{-1} C_t$ is found by eliminating x algebraically to give an equation in x_0.

When $\Phi_t^{-1}\{x\} = \{x_0(1)(x,t), x_0(2)(x,t), \ldots, x_0(n)(x,t)\}$, where each $x_0(i)(x,t) \in \mathbb{R}^d$, the Feynman–Kac formula and Laplace's method in infinite dimensions give for a non-degenerate critical point:

$$u^\mu(x,t) = \sum_{i=1}^{n} \theta_i \exp\left(-\frac{S_0^i(x,t)}{\mu^2}\right),$$

where

$$S_0^i(x,t) = S_0(x_0(i)(x,t)) + A(x_0(i)(x,t), x, t),$$

θ_i being an asymptotic series in μ^2.

When $x_0(x,t) = x_0(1)(x,t)$ is unique, $t < T(\omega)$, Hamilton–Jacobi theory [6] gives for each integer m:

$$v^\mu(x,t) = \sum_{j=0}^{m} \mu^{2j} v_j(x,t) - \mu^2 \nabla \ln \mathbb{E}\left\{\exp\left(\frac{-\mu^{2m}}{2}\int_0^t \nabla \cdot v_m\big(y_s^\mu, t-s\big)\,\mathrm{d}s\right.\right.$$
$$\left.\left. + \frac{1}{2}\sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0 \leqslant i_1 \leqslant i_2 \leqslant m \\ i_1+i_2=j}} \int_0^t v_{i_1} \cdot v_{i_2}\big(y_s^\mu, t-s\big)\,\mathrm{d}s\right)\right\},$$

where $v_j(x,t) = \nabla S_j(x,t)$ and S_j satisfies the Hamilton–Jacobi continuity equations

$$\frac{\partial S_j}{\partial t} + \frac{1}{2}\sum_{\substack{i_1,i_2 \geqslant 0 \\ i_1+i_2=j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2}\Delta S_{j-1},$$

for $j = 0, 1, 2, \ldots$, with the convention $\frac{1}{2}\Delta S_{-1} = -V - \epsilon k_t \dot{W}_t$. Here each S_j can be found explicitly.

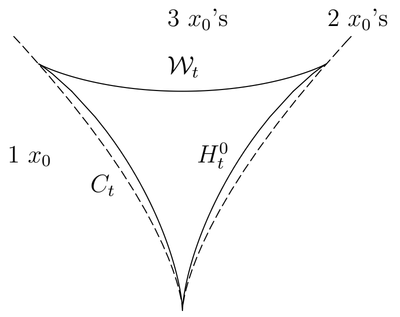

Figure 1. Cusp and tricorn.

Moreover, y_s^μ, the Nelson diffusion process, is given by

$$\mathrm{d}y_s^\mu = \mu\,\mathrm{d}B_s - \nabla\sum_{j=0}^{m}\mu^{2j}S_j\left(y_s^\mu, t-s\right)\mathrm{d}s, \qquad y_0^\mu = x, \quad y_t^\mu = x_0(x,t).$$

Therefore, the asymptotic series θ is known explicitly and as $\mu \sim 0$,

$$v^\mu(x,t) \sim \nabla S_0(x,t) + \mathrm{O}(\mu^2),$$

$S_0(x,t)$ being the *minimizing action* as expected.

When $\Phi_t^{-1}\{x\} = \{x_0(1)(x,t), x_0(2)(x,t), \ldots, x_0(n)(x,t)\}$, there is a similar asymptotic series θ_i to the above corresponding to $x_0(i)(x,t)$, but in this case there is no simple form for the remainder term. Observe that $\mathcal{S}(x,t) = \min_{i=1,2,\ldots,n} S_0^i(x,t)$.

Definition 1.3. *The Hamilton–Jacobi level surfaces are defined by the equation*

$$H_t^c = \left\{x : S_0^i(x,t) = c \text{ for some } i\right\},$$

where H_t^0 includes the wavefront $\mathcal{W}_t$. The pre-level surface $\Phi_t^{-1}H_t^c$ is the expression in x_0 found by algebraically eliminating x using $x = \Phi_t(x_0)$.

The dominant term for $v^0(x,t)$ comes from the minimizing $x_0(i)(x,t)$ denoted by $\tilde{x}_0(x,t)$ (assumed unique) and we obtain the corresponding inviscid limit of the Burgers fluid velocity:

$$v^0(x,t) = \dot{\Phi}_t\tilde{x}_0(x,t).$$

Two $x_0(i)(x,t)$'s can coalesce and disappear (become complex), as we cross the caustic surface C_t. When this corresponds to the minimizing $x_0(i)(x,t)$ jumping, $u^{\mu=0}$ and $v^{\mu=0}$ have jump discontinuities, and we describe the caustic as cool. Alternatively, we can have a jump if there are two distinct minimizers returning the same minimum value of the action. Such points are said to be in the cool part of the Maxwell set.

Example 1.4 (the generic cusp)**.** Let $V(x,y) = 0$, $k_t(x,y) = 0$, and $S_0(x_0,y_0) = x_0^2 y_0/2$. This initial condition leads to the *generic cusp*, a semicubical parabolic caustic. This is shown in figure 1. The caustic C_t (dashed line) is given by

$$x_t(x_0) = t^2x_0^3, \qquad y_t(x_0) = \frac{3}{2}tx_0^2 - \frac{1}{t}.$$

The zero level surface H_t^0 (solid line) is

$$x_{(t,0)}(x_0) = \frac{x_0}{2}\left(1 \pm \sqrt{1-t^2x_0^2}\right), \qquad y_{(t,0)}(x_0) = \frac{1}{2t}\left(t^2x_0^2 - 1 \pm \sqrt{1-t^2x_0^2}\right).$$

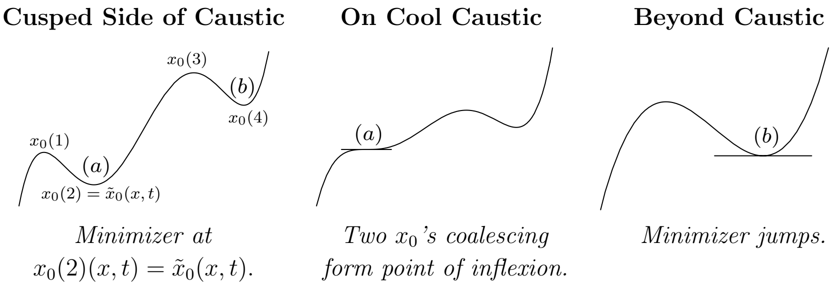

Figure 2. The graph of the phase function as x crosses the caustic.

Evidently n, the multiplicity of real $x_0(x,t)$'s, depends upon x and t. This multiplicity changes by multiples of 2 as we cross the caustic surface. It can be shown that this is associated with level surfaces of Hamilton's characteristic function having cusps on the caustic (see figure 1).

We illustrate this in one dimension by considering the integral

$$I(x,t) = \int_{\mathbb{R}} G(x_0) \exp\left(\mathrm{i}\frac{F(x_0,x,t)}{\mu^2}\right) \mathrm{d}x_0,$$

where $G \in C_0^\infty(\mathbb{R})$ and $\mathrm{i} = \sqrt{-1}$ for small μ. Consider the graph of the phase function, $F_{(x,t)}(x_0) = F(x_0,x,t)$, as x crosses the caustic surface C_t (see figure 2).

Here $\tilde{x}_0(x,t)$ jumps from (a) to (b) causing u^μ and v^μ for small μ to have jump discontinuities. This only occurs when the point of inflexion is the global minimizer of F. When two x_0's coalesce at a minimum of F, we are on a 'cool part of the caustic' giving jump discontinuities in u^μ and v^μ for small μ. Two x_0's coalescing and becoming complex, correspond to the level surface H_t^c having cusps as indicated in figures 3 and 4.

(Note that the cusped part of the level surface, $\mathrm{Cusp}(H_t^c) = \Phi_t(\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t^c)$ and that normally $\Phi_t^{-1}H_t^c \neq \Phi_t^{-1}(H_t^c)$, where $\Phi_t^{-1}(H_t^c)$ denotes the topological inverse image, while $\Phi_t^{-1}H_t^c$ is determined algebraically by eliminating x between $\mathcal{A}(x_0,x,t) = c$ and $x = \Phi_t(x_0)$.)

Cusps occur on the level surfaces of Hamilton's characteristic function at points where the corresponding pre-level surfaces intersect the pre-caustic. In three dimensions, let $n_c(t)$ be the number of cusped curves in $C_t \cap H_t^c$, or in two dimensions, the number of cusps on H_t^c. We are interested in when $n_c(t)$ changes. The simplest way for this to happen is for the corresponding pre-surface to touch the pre-caustic. The times t when $n_c(t)$ changes are the zeros of a stochastic process ζ (zeta), i.e. times t when $\zeta(t) = 0$. When the driving force is white noise in time, such times t form a perfect set, i.e. an infinite set with no isolated points. Thus, at such times the geometry of the level surface of the Hamilton–Jacobi function changes infinitely rapidly as it would in turbulent behaviour. We call this phenomenon *real stochastic turbulence*.

There is another kind of *complex stochastic turbulence* associated with the infinitely fast creation and destruction of tiny swallowtails on the caustic determined by the zeros of the resultant eta process ρ_η. This occurs when the real pre-caustic touches its complex counterpart. The tiny swallowtails formed in this way contain Maxwell sets across which typically v^0 is discontinuous.

When we cross the Maxwell set, two critical points of the phase function F return the same F value. If these correspond to the minimizing x_0, our solutions for u^0 and v^0 will jump (figure 11). This event is equivalent to the level surfaces of the Hamilton–Jacobi function having a point of self-intersection, and clearly can only occur in regions with three or more

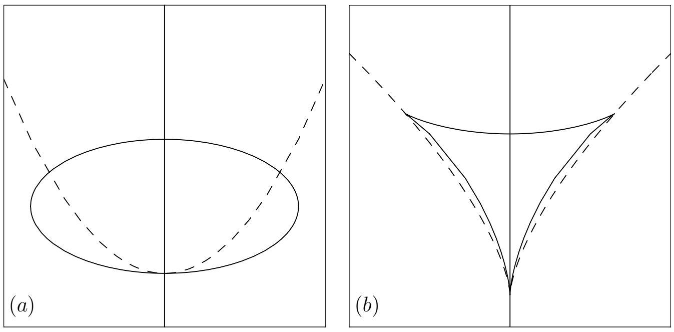

Figure 3. (a) The zero pre-level surface (solid line) and pre-caustic (dashed line), and (b) the zero level surface (solid line) and caustic (dashed line), both for the generic cusp.

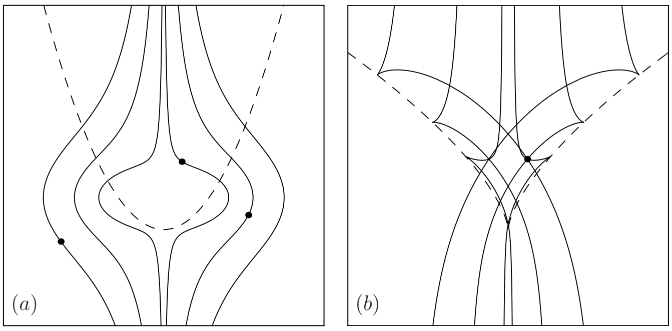

Figure 4. (a) The pre-level surface (solid line) and pre-caustic (dashed line), and (b) the level surface (solid line) and caustic (dashed line), both for the generic cusp with $c > 0$.

pre-images under Φ_t. As we shall see, in the two-dimensional case, this provides a fundamental link between the existence of the Maxwell set and swallowtails on the caustic and level surfaces. Hence, these play a role in the special nature of complex turbulence.

We begin, in section 2, by recapitulating some results from [7]. Section 3 then develops the one-dimensional analysis which we use extensively throughout our results. Section 4 contains a detailed analysis of singularities in two dimensions. We investigate the Maxwell set by considering the Maxwell–Klein set of non-cusp double points of the level surface. As we shall show, this can be found easily as a result of a simple factorization (theorems 4.15 and 4.16). We then demonstrate, in section 6, how these ideas can be extended to find the set of all discontinuities for the inviscid limit of the Burgers fluid velocity in any dimension (theorem 6.3). This solves a long-standing problem in applied mathematics and yields the Maxwell set as part of an algebraic surface.

Section 5 discusses the nature of the two-dimensional turbulence and gives explicit formulae for the cusp density on level surfaces and the caustic. We shall see that real turbulence can only occur at points in the cool part of the caustic where the caustic tangent has a scalar product zero with the Burgers fluid velocity. We show that the ζ process is just the reduced action function evaluated at these specific points on the caustic. The recurrence of ζ is then equivalent to the stochastic turbulence being intermittent. It is shown that complex turbulence is a special form of real turbulence occurring at certain generalized cusps of the caustic. Complex turbulence can only occur at times t which are the zeros of ρ_η, the resultant eta process given by the discriminant of the third derivative of the reduced action function evaluated on the caustic.

The results in this paper are valid for the stochastic Burgers equation but also hold in the case of zero noise ($\epsilon = 0$), i.e. for the deterministic Burgers equation. In some examples we work exclusively with zero noise, but this is explicitly stated. The turbulent phenomena outlined above can only arise in the stochastic case. Although there are equivalent results for the deterministic case, these do not give rise to turbulent behaviour.

Notation. Throughout this paper $x, x_0, x_t, x_{(t,c)}$, etc will denote vectors, where normally $x = \Phi_t x_0$. Cartesian coordinates of these will be indicated using a sub/superscript where relevant, so $x = (x_1, x_2, \ldots, x_d)$, $x_0 = \left(x_0^1, x_0^2, \ldots, x_0^d\right)$, etc. The only exception to this will be when we discuss explicit examples in two and three dimensions when we shall use (x, y), (x_0, y_0), etc to denote these vectors.

2. Geometrical results of DTZ in two dimensions (or more)

We summarize the geometrical relationships between curves which are level surfaces of the Hamilton–Jacobi function and shockwaves or caustics for the Burgers equation as established by Davies, Truman and Zhao (DTZ) [7].

Definition 2.1. *A curve $x = x(\gamma)$, $\gamma \in N(\gamma_0, \delta)$, is said to have a generalized cusp at $\gamma = \gamma_0$, γ being an intrinsic variable such as arc length, if*

$$\left.\frac{\mathrm{d}x}{\mathrm{d}\gamma}\right|_{\gamma=\gamma_0} = 0.$$

We begin by considering the deterministic case, where $x_0 = \left(x_0^1, x_0^2, \ldots, x_0^d\right)$ and $x = (x_1, x_2, \ldots, x_d)$ so that $x, x_0 \in \mathbb{R}^d$ and $t > 0$:

$$\mathcal{A} = \mathcal{A}(x_0, x, t) = S_0(x_0) + A(x_0, x, t),$$

where

$$A(x_0, x, t) = \inf_{\substack{X(0)=x_0 \\ X(t)=x}} \left[\int_0^t \left\{\frac{1}{2}\dot{X}(s)^2 - V(X(s))\right\} \mathrm{d}s\right].$$

The corresponding Euler Lagrange equation reads

$$\ddot{X}(s) = -\nabla V(X(s)), \qquad s \in [0, t], \qquad X(t) = x, \qquad X(0) = x_0.$$

In the free case, $V \equiv 0$ so

$$\mathcal{A}(x_0, x, t) = \frac{(x - x_0)^2}{2t} + S_0(x_0).$$

We assume that $\mathcal{A}(x_0, x, t)$ is C^4 in space variables for times $t > 0$, giving

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha} = 0, \qquad \alpha = 1, 2, \ldots, d \quad \Leftrightarrow \quad x = \Phi_t x_0 = x_0 + t\nabla S_0(x_0).$$

We shall see later that this result is true in enormous generality.

The Hamilton–Jacobi level surface H_t^c is obtained by eliminating x_0 between

$$\mathcal{A}(x_0, x, t) = c \qquad \text{and} \qquad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0, \qquad \alpha = 1, 2, \ldots, d.$$

Alternatively, eliminating x gives the pre-level surface $\Phi_t^{-1}H_t^c$. Similarly, the pre-caustic $\Phi_t^{-1}C_t$ (and caustic C_t) are obtained by eliminating x (or x_0) between

$$\det\left(\frac{\partial^2\mathcal{A}}{\partial x_0^\alpha\partial x_0^\beta}(x_0,x,t)\right)_{\alpha,\beta=1,2,\ldots,d}=0 \qquad \text{and} \qquad \frac{\partial\mathcal{A}}{\partial x_0^\alpha}(x_0,x,t)=0,$$
$$\text{for}\,\alpha=1,2,\ldots,d.$$

In the free case, the equation for the zero pre-level surface is the *eikonal equation*

$$\frac{t}{2}|\nabla S_0(x_0)|^2+S_0(x_0)=0,$$

and the derivative map $D\Phi_t(x_0)$ is given by the Hessian

$$D\Phi_t(x_0)=I+t\nabla^2S_0(x_0).$$

These, together with the key identity

$$\nabla_{x_0}\left\{\frac{t}{2}|\nabla S_0(x_0)|^2+S_0(x_0)\right\}=(1+t\nabla^2S_0(x_0))\nabla S_0(x_0),$$

give us the following results.

Lemma 2.2 (free case in two dimensions)**.** *Assume that the pre-level surface meets the pre-caustic at x_0 where $|(I+t\nabla^2S_0(x_0))\nabla S_0(x_0)|\neq 0$ and $\dim(\ker(I+t\nabla^2S_0(x_0)))=1$. Then, the tangent plane to the pre-level surface T_{x_0} is spanned by $\ker(I+t\nabla^2S_0(x_0))$.*

Proposition 2.3 (free case in two dimensions)**.** *Assume that $|(I+t\nabla^2S_0(x_0))\nabla S_0(x_0)|\neq 0$, so that x_0 is not a singular point of $\Phi_t^{-1}H_t^c$, then $\Phi_t x_0$ can only be a generalized cusp if $\Phi_t(x_0)\in C_t$, the caustic. Moreover, if $x=\Phi_t x_0\in\Phi_t\left(\Phi_t^{-1}C_t\cap\Phi_t^{-1}H_t^c\right)$, x will indeed be a generalized cusp of the level surface.*

The above results generalize to d dimensions, to very general deterministic systems and to systems with noise. Let the stochastic action be defined by

$$A(x_0,p_0,t)=\frac{1}{2}\int_0^t\dot{X}(s)^2\,\mathrm{d}s-\int_0^t\left[V(X(s))\,\mathrm{d}s+\epsilon k_s(X(s))\,\mathrm{d}W_s\right],$$

where $X_s=X(s)=X(s,x_0,p_0)\in\mathbb{R}^d$ and

$$\mathrm{d}\dot{X}(s)=-\nabla V(X(s))\,\mathrm{d}s-\epsilon\nabla k_s(X(s))\,\mathrm{d}W_s, \qquad s\in[0,t],$$

with $X(0)=x_0$, $\dot{X}(0)=p_0$ and $x_0,p_0\in\mathbb{R}^d$. We assume X_s is $\mathcal{F}_s$ measurable and unique. If $\mathrm{d}u_s\,\mathrm{d}\dot{X}(s)=0$, we have

$$\int_0^t u(s)\,\mathrm{d}\dot{X}(s)=u(t)\dot{X}(t)-u(0)\dot{X}(0)-\int_0^t\dot{u}(s)\dot{X}(s)\,\mathrm{d}s.$$

In particular, this is true when $u_s=\frac{\partial X_s}{\partial x_0^\alpha}$, where $\alpha=1,2,\ldots,d$. Using Kunita [8], mild regularity gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial X_s}{\partial x_0^\alpha}\right)=\frac{\partial\dot{X}_s}{\partial x_0^\alpha}, \qquad \alpha=1,2,\ldots,d,$$

almost surely.

Lemma 2.4 (d dimensions)**.** *Assume $S_0, V \in C^2$ and $k \in C^{2,0}$, $\nabla V, \nabla k$ Lipschitz with Hessians $\nabla^2 V, \nabla^2 k$ and all second derivatives with respect to space variables of V and k bounded. Then, for p_0 possibly x_0 dependent,*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}_\alpha(0), \qquad \alpha = 1, 2, \ldots, d.$$

Now let

$$A(x_0, x, t) = A(x_0, p_0, t)|_{p_0 = p_0(x_0, x, t)},$$

where $p_0 = p_0(x_0, x, t)$ is the random minimizer (assumed unique) of $A(x_0, p_0, t)$ when $X(t, x_0, p_0) = x$. (Here we need the map $p_0 \mapsto X(t, x_0, p_0) \in \mathbb{R}^d$ to be onto for all x_0. Methods of Kolokoltsov *et al* [9] guarantee this for small t.)

Theorem 2.5 (d dimensions)**.** *Let the stochastic flow map be Φ_t, then $x = \Phi_t(x_0)$ is equivalent to*

$$\frac{\partial}{\partial x_0^\alpha}[S_0(x_0) + A(x_0, x, t)] = 0, \qquad \alpha = 1, 2, \ldots, d.$$

We now define the stochastic action corresponding to the initial momentum $\nabla S_0(x_0)$ by

$$\mathcal{A}(x_0, x, t) = A(x_0, x, t) + S_0(x_0).$$

Assume that $A(x_0, x, t)$ is C^4 in space variables with $\det\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha \partial x^\beta}\right) \neq 0$, and that $\ker(D\Phi_t)$ is one dimensional. Then, we have

Proposition 2.6 (d dimensions)**.** *The random classical flow map has Frechet derivative a.s.*

$$(D\Phi_t)(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1} \left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_0}(x_0, x, t)\right)$$

and the normal to the pre-level surface is

$$n(x_0) = -\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_0}\right)\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1} \dot{X}(t, x_0, \nabla S_0(x_0)).$$

(These are analogues of

$$D\Phi_t(x_0) = (I + t\nabla^2 S_0(x_0))$$

and

$$\nabla_{x_0}\left\{\frac{t}{2}|\nabla S_0(x_0)|^2 + S_0(x_0)\right\} = (I + t\nabla^2 S_0(x_0))\nabla S_0(x_0)$$

in the free case.)

We content ourselves with quoting

Theorem 2.7 (three dimensions)**.** *Let*

$$x \in Cusp\left(H_t^c\right) = \left\{x \in \Phi_t\left(\Phi_t^{-1} C_t \cap \Phi_t^{-1} H_t^c\right), x = \Phi_t(x_0), n(x_0) \neq 0\right\}.$$

Then, in three dimensions in the stochastic case, with probability 1, T_x the tangent space to the level surface at x is at most one dimensional.

3. A one-dimensional analysis

3.1. Global reducibility

We now explain how a one-dimensional analysis first described by Reynolds, Truman and Williams (RTW) [10] greatly simplifies our approach.

Definition 3.1. *The classical flow map Φ_t is globally reducible if*

$$x = \Phi_t x_0, \qquad x = (x_1, x_2, \ldots, x_d), \qquad x_0 = \left(x_0^1, x_0^2, \ldots, x_0^d\right)$$
$$\Rightarrow \quad x_0^\alpha = x_0^\alpha\left(x, x_0^1, x_0^2, \ldots, x_0^{\alpha-1}, t\right), \qquad \alpha = d, d-1, \ldots, 2. \tag{1}$$

We want C^2 functions $x_0^d, x_0^{d-1}, \ldots, x_0^2$ such that

$$x_0^d = x_0^d\left(x, x_0^1, x_0^2, \ldots, x_0^{d-1}, t\right) \quad \Leftrightarrow \quad \frac{\partial \mathcal{A}}{\partial x_0^d}(x_0, x, t) = 0,$$
$$x_0^{d-1} = x_0^{d-1}\left(x, x_0^1, x_0^2, \ldots, x_0^{d-2}, t\right) \quad \Leftrightarrow \quad \frac{\partial \mathcal{A}}{\partial x_0^{d-1}}\left(x_0^1, x_0^2, \ldots, x_0^d(\cdot), x, t\right) = 0,$$
$$\vdots$$
$$x_0^2 = x_0^2\left(x, x_0^1, t\right) \quad \Leftrightarrow \quad \frac{\partial \mathcal{A}}{\partial x_0^2}\left(x_0^1, x_0^2, x_0^3\left(x, x_0^1, x_0^2, t\right), \ldots, x_0^d(\cdot), x, t\right) = 0,$$

where $x_0^d(\cdot) = x_0^d\left(x, x_0^1, x_0^2, \ldots, x_0^{d-1}, t\right)$. We assume that no root is repeated so the second derivatives of $\mathcal{A}$ do not vanish. (Some local reducibility follows from mild assumptions about $\mathcal{A}$ and its derivatives.) Evidently, we are assuming a favoured ordering of coordinates and a corresponding decomposition of Φ_t allowing the non-uniqueness to be reduced to the level of the x_0^1 coordinate. We begin with a result of RTW.

Proposition 3.2. *Assume Φ_t is globally reducible. Define the reduced action*

$$f_{(x,t)}\left(x_0^1\right) := f\left(x_0^1, x, t\right) = \mathcal{A}\left(x_0^1, x_0^2\left(x, x_0^1, t\right), x_0^3(\cdot), \ldots, x, t\right).$$

Then,

(i) $\frac{\partial f}{\partial x_0^1}\left(x_0^1, x, t\right) = 0$ *and the previous equation (1)* $\Leftrightarrow x = \Phi_t x_0$*;*

(ii) $\frac{\partial f}{\partial x_0^1}\left(x_0^1, x, t\right) = \frac{\partial^2 f}{\left(\partial x_0^1\right)^2}\left(x_0^1, x, t\right) = 0$ *and the previous equation (1)* $\Leftrightarrow x = \Phi_t x_0$ *is such that the number of real solutions x_0 of this equation changes.*

The key to this proof is the following lemma.

Lemma 3.3. *If Φ_t is globally reducible,*

$$\left|\det\left(\frac{\partial^2 \mathcal{A}}{(\partial x_0)^2}(x_0, x, t)\right)\bigg|_{x=\Phi_t x_0}\right| = \prod_{\alpha=0}^{d-1}\left|\frac{\partial^2 \mathcal{A}}{\left(\partial x_0^{d-\alpha}\right)^2}\left(x_0^1, x_0^2\left(x, x_0^1, t\right), \ldots, x_0^d(\cdot), x, t\right)\right|,$$

where the last term is $f''_{(x,t)}\left(x_0^1\right)$ and the first $d-1$ terms are non-zero.

Proof. The principle of stationary phase applied to the evaluation of

$$I = \int_{\mathbb{R}^d} G(x_0) \exp\left(-\frac{\mathrm{i}}{\mu^2}\mathcal{A}(x_0, x, t)\right) \mathrm{d}x_0,$$

by repeated integration shows that, if we assume

$$\frac{\partial f}{\partial x_0^1}\left(x_0^1, x, t\right) = 0$$

has n real roots and that x is such that

$$\frac{\partial^2 f}{\left(\partial x_0^1\right)^2}\left(x_0^1, x, t\right) \neq 0,$$

then the first equation will have n real simple roots, critical points of f corresponding to $\{x_0(i)(x,t)\}_{i=1,2,\ldots,n}$. Varying x so that

$$\frac{\partial^2 f}{\left(\partial x_0^1\right)^2}\left(x_0^1, x, t\right) = 0,$$

allows typically two of these n critical points to coalesce. □

The phase function F in the introduction is simply this reduced action function $F\left(x_0^1, x, t\right) = f\left(x_0^1, x, t\right)$ where $x \in \mathbb{R}^d$. From here we denote $f\left(x_0^1, x, t\right) = f_{(x,t)}\left(x_0^1\right)$ to highlight its univariance in x_0^1. Also note that if $x_0(i)(x,t)$ denote the real pre-images of x, then

$$\begin{aligned} x_0(i)(x,t) &:= \left(x_0^1(i)(x,t), x_0^2(i)(x,t), \ldots, x_0^d(i)(x,t)\right) \\ &= \left(x_0^1(i)(x,t), x_0^2\left(x, x_0^1(i)(x,t), t\right), \ldots, x_0^d\left(x, x_0^1(i)(x,t), \ldots, t\right)\right), \end{aligned}$$

where $x_0^1(i)(x,t)$ is then an enumeration of the real roots x_0^1 of $f'_{(x,t)}\left(x_0^1\right) = 0$.

This reduced action function provides a universal one-dimensional analysis of all aspects of the problem. The caustic surface is found by eliminating the x_0^1 variable between

$$f'_{(x,t)}\left(x_0^1\right) = 0 \qquad \text{and} \qquad f''_{(x,t)}\left(x_0^1\right) = 0.$$

This allows us to view the caustic as the bifurcation set of the univariate function f. The level surfaces are found by eliminating x_0^1 between

$$f_{(x,t)}\left(x_0^1\right) = c \qquad \text{and} \qquad f'_{(x,t)}\left(x_0^1\right) = 0.$$

For polynomial f, these eliminations can be done simply by taking resultants with respect to x_0^1. This is found by taking the determinant of the Sylvester matrix [11]. We shall return to this point in section 6.3. The reduced action also provides information on the geometry of the caustic, level surfaces and Maxwell set, and, as we shall explain, gives a simple approach to points of turbulent behaviour on the caustic.

3.2. Caustic parametrization

Corollary 3.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1}$ where $\lambda \mapsto \left(\lambda, x_{0,\mathrm{C}}^d(\lambda)\right)$ is a parametrization of the pre-caustic at time t, so that $x_t(\lambda) = \Phi_t\left(\lambda, x_{0,\mathrm{C}}^d(\lambda)\right)$ is a parametrization of the caustic—the pre-parametrization. Then,*

$$f'_{(x_t(\lambda),t)}(\lambda_1) = f''_{(x_t(\lambda),t)}(\lambda_1) = 0.$$

Proof. A simple corollary to proposition 3.2 using the fact that $x_t(\lambda) = \Phi_t\left(\lambda, x_{0,\mathrm{C}}^d(\lambda)\right)$. □

Thus, there is a critical point of inflexion on f at $x_0^1 = \lambda_1$ for this parametrization. Moreover, the geometry of the caustic is determined by the higher derivatives of f. In d dimensions we call a part of the caustic with $(d-2)$-dimensional tangent space associated with the above pre-parametrization the *principal subcaustic*,

$$x_t^{\mathrm{sc}} = \Phi_t\big(\lambda, x_{0,\mathrm{C}}^d(\lambda)\big)\big|_{\lambda_{d-1}=\lambda_{d-1}(\lambda_1,\ldots,\lambda_{d-2})},$$

where $\lambda_{d-1}(\cdot)$ is determined by

$$\det_{\alpha,\beta=1,\ldots,d-1}\left(\frac{\partial x_t}{\partial\lambda_\alpha}\cdot\frac{\partial x_t}{\partial\lambda_\beta}\right) = 0.$$

In two dimensions this corresponds to cusps and in three dimensions to creasing the caustic along a curve. Here we must assume that the derivatives $\left\{\frac{\partial\Phi_t(x_0)}{\partial x_0^\alpha}\right\}_{\alpha=2,3,\ldots,d}$ are always linearly independent.

Proposition 3.5. *If, using the above pre-parametrization, $x_t(\lambda)$ is on the subcaustic, then*

$$f'_{(x_t(\lambda),t)}(\lambda_1) = f''_{(x_t(\lambda),t)}(\lambda_1) = f'''_{(x_t(\lambda),t)}(\lambda_1) = 0.$$

Proof. Differentiating the equation $f''_{(x_t(\lambda),t)}(\lambda_1) = 0$ with respect to each λ_α gives the $d-1$ equations

$$0 = \frac{\partial x_t}{\partial\lambda_1}(\lambda)\cdot\nabla_x f''_{(x_t(\lambda),t)}(\lambda_1) + f'''_{(x_t(\lambda),t)}(\lambda_1),$$
$$0 = \frac{\partial x_t}{\partial\lambda_\alpha}(\lambda)\cdot\nabla_x f''_{(x_t(\lambda),t)}(\lambda_1),$$

where $\alpha = 2,\ldots,d-1$. On the subcaustic, there exist scalars κ_α with $\kappa_1 \neq 0$ such that

$$\sum_{\alpha=1}^{d-1}\kappa_\alpha\frac{\partial x_t}{\partial\lambda_\alpha} = 0,$$

and the result follows. □

We can also form the converse to this proposition. For instance, in the two-dimensional case:

Theorem 3.6. *If in two dimensions*

$$f'_{(x_t(\lambda_0),t)}(\lambda_0) = f''_{(x_t(\lambda_0),t)}(\lambda_0) = f'''_{(x_t(\lambda_0),t)}(\lambda_0) = 0,$$

and the vectors

$$\nabla_x f'_{(x_t(\lambda_0),t)}(\lambda_0), \qquad \nabla_x f''_{(x_t(\lambda_0),t)}(\lambda_0)$$

are linearly independent, then there is a generalized cusp on the caustic at $x_t(\lambda_0)$.

Proof. As we are dealing with two dimensions, we have no need for an index for λ or λ_0. Differentiating the two equations $f'_{(x_t(\lambda),t)}(\lambda) = f''_{(x_t(\lambda),t)}(\lambda) = 0$ with respect to λ yields

$$x_t'(\lambda)\cdot\nabla_x f'_{(x_t(\lambda),t)}(\lambda) = 0, \qquad x_t'(\lambda)\cdot\nabla_x f''_{(x_t(\lambda),t)}(\lambda) + f'''_{(x_t(\lambda),t)}(\lambda) = 0.$$

Setting $\lambda = \lambda_0$, the result follows. □

This idea can be extended to higher derivatives of the reduced action function, for instance:

Proposition 3.7. *In three dimensions, if there is a generalized cusp on the subcaustic at a point $x_t(\lambda)$ then*

$$f'_{(x_t(\lambda),t)}(\lambda_1) = f''_{(x_t(\lambda),t)}(\lambda_1) = f'''_{(x_t(\lambda),t)}(\lambda_1) = f^{(4)}_{(x_t(\lambda),t)}(\lambda_1) = 0.$$

We can characterize other features of the caustic in terms of the reduced action function. For instance,

Lemma 3.8. *There is a point of self-intersection at $x_t(\lambda)$ if and only if there exists a solution*

$$f'_{(x_t(\lambda),t)}\left(x_0^1\right) = f''_{(x_t(\lambda),t)}\left(x_0^1\right) = 0,$$

with $x_0^1 \neq \lambda_1$.

Proof. A simple result following from proposition 3.2. □

3.3. Hot and cool caustics

The hot and cool designation divides the caustic into two distinct regions where the inviscid limit of the Burgers fluid velocity is either continuous (hot) or discontinuous (cool) (recall figure 2).

Definition 3.9. *Let $x_t(\lambda)$ be the pre-parametrization of the caustic using the pre-caustic. Then, $x_t(\lambda)$ is on the cool part of the caustic if $f_{(x_t(\lambda),t)}(\lambda_1) \leqslant f_{(x_t(\lambda),t)}\left(x_0^1(i)(x_t(\lambda),t)\right)$ for all $i = 1, 2, \ldots, n$, where $x_0^1(i)(x,t)$ denotes an enumeration of all the real roots for x_0^1 to $f'_{(x,t)}\left(x_0^1\right) = 0$. If the caustic is not cool, it is hot.*

Definition 3.10. *The pre-normalized reduced action function evaluated on the caustic is given by*

$$\mathcal{F}_\lambda\left(x_0^1\right) = f_{(x_t(\lambda),t)}\left(x_0^1\right) - f_{(x_t(\lambda),t)}(\lambda_1).$$

Assume that $\mathcal{F}_\lambda\left(x_0^1\right)$ is a real analytic function in a neighbourhood of $\lambda_1 \in \mathbb{R}$. Clearly, $\mathcal{F}_\lambda(\lambda_1) = \mathcal{F}'_\lambda(\lambda_1) = \mathcal{F}''_\lambda(\lambda_1) = 0$, and so

$$\mathcal{F}_\lambda\left(x_0^1\right) = \left(x_0^1 - \lambda_1\right)^3 \tilde{F}\left(x_0^1\right),$$

where $\tilde{F}$ is real analytic. When the inflexion at $x_0^1 = \lambda_1$ is the minimizing critical point of $\mathcal{F}_\lambda$, the caustic will be cool. Therefore, on a hot/cool boundary, this inflexion is about to become or cease being the minimizer.

Proposition 3.11. *A necessary condition for $x_t(\lambda) \in C_t$ to be on a hot/cool boundary is that either*

(i) $\tilde{F}\left(x_0^1\right)$ *or*
(ii) $\tilde{G}\left(x_0^1\right) = 3\tilde{F}\left(x_0^1\right) + \left(x_0^1 - \lambda_1\right)\tilde{F}'\left(x_0^1\right)$

has a repeated root at $x_0^1 = r$. Note that normally $r \neq \lambda^1$.

Proof. There are two ways in which the minimizer could change. Firstly, $\tilde{F}$ could have a repeated root which is also the minimizer. Secondly, there could be another inflexion at a lower value which is the minimizer. This would correspond to a point of self-intersection of the caustic. □

The condition is not sufficient as it includes both cases where the repeated root is complex and where the repeated root is real but there is not a hot/cool boundary (see figure 5).

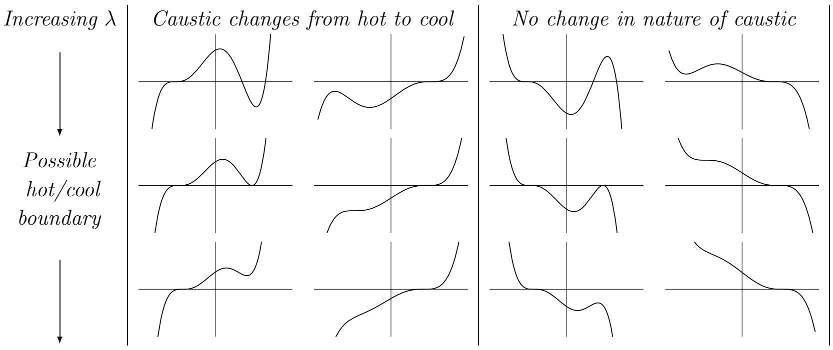

Figure 5. Graphs of $\mathcal{F}_\lambda(x_0^1)$.

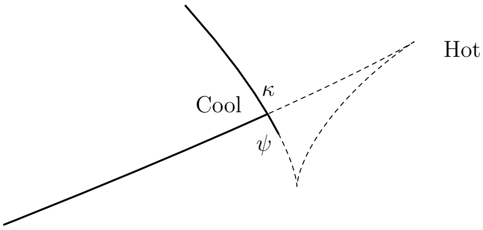

Figure 6. Hot and cool parts of the polynomial swallowtail caustic at time $t = 1$.

Example 3.12 (the polynomial swallowtail)**.** Let $V(x, y) \equiv 0$, $k_t(x, y) \equiv x$ and

$$S_0(x_0, y_0) = x_0^5 + x_0^2 y_0.$$

This gives global reducibility and $k_t(x, y) \equiv x$ means that the effect of the noise is to translate the $\epsilon = 0$ picture through $\left(-\epsilon \int_0^t W_s \,\mathrm{d}s, 0\right)$. A simple calculation gives

$$\tilde{F}(x_0) = 12\lambda^2 - 3\lambda t + 6\lambda x_0 - t x_0 + 2x_0^2,$$
$$\tilde{G}(x_0) = 15\lambda^2 - 4\lambda t + 10\lambda x_0 - 2t x_0 + 5x_0^2,$$

so that the boundary points are

$$\kappa = \left(-\frac{t^5}{500} - \epsilon \int_0^t W_s \,\mathrm{d}s, -\frac{1}{2t} + \frac{t^3}{50}\right),$$
$$\psi = \left(-\frac{t^5(3 + 8\sqrt{6})}{18\,000} - \epsilon \int_0^t W_s \,\mathrm{d}s, -\frac{1}{2t} + \frac{t^3(9 - \sqrt{6})}{450}\right).$$

This is illustrated in figure 6 where the cool parts are shown with a thick solid line and the hot parts with a thin dashed line. Note that κ is the point of self intersection (crunode) and so arises from $\tilde{G}$ whereas ψ is a regular point of the caustic and arises from $\tilde{F}$.

4. Further geometric results in two dimensions

It is well known that the geometry of a caustic curve or wavefront can suddenly change with singularities appearing and disappearing. Arnol'd classified six such perestroikas for two-dimensional caustics and five for wavefronts [12]. Using the much earlier work of Cayley

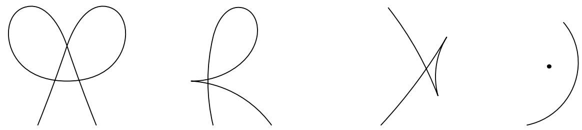

Figure 7. Cayley's triple points.

and Klein, we investigate one of these perestroikas, namely the formation or collapse of a swallowtail.

In his work on plane algebraic curves, Cayley described the possible triple points of a curve [13]. He classified these by considering the collapse of systems of double points which would lead to the existence of three tangents at a point. The four possibilities are shown in figure 7 where the systems will collapse to form a triple point with

(i) three real distinct tangents;
(ii) three real tangents with two coincident;
(iii) three real tangents all of which are coincident;
(iv) one real tangent and two complex tangents.

We are particularly interested in the possibility of interchange between the last two cases. Felix Klein proved in his work on Riemann surfaces that a swallowtail forms when an isolated double point (acnode) joins the main curve [14].

Previously, we have parametrized our curves and allowed our parameter to vary only through the reals. Thus, we do not normally obtain isolated double points. However, swallowtails do still form on some of these curves.

We find the isolated double points by allowing our parameter to vary throughout the complex plane and then considering when this maps to real points. We assume that the curves are of the form $x(\lambda) = (x_1(\lambda), x_2(\lambda))$ where each $x_\alpha(\lambda)$ is real analytic in $\lambda \in \mathbb{C}$ so that if $\lambda = a + \mathrm{i}\eta$ where $a, \eta \in \mathbb{R}$,

$$\overline{x(a+\mathrm{i}\eta)} = x(a-\mathrm{i}\eta).$$

Then, if $\mathrm{Im}\{x(a+\mathrm{i}\eta)\} = 0$,

$$x(a+\mathrm{i}\eta) = x(a-\mathrm{i}\eta),$$

so such a point is a double point. We will refer to these points as '*complex double points*' of the curve $x(\lambda)$.

We can interpret the definition of generalized cusps in terms of complex parameter values by following a simple idea of Klein.

Lemma 4.1. *If $x(\lambda) = (x_1(\lambda), x_2(\lambda))$ is a real analytic parametrization of a curve and λ is an intrinsic parameter, then there is a generalized cusp at $\lambda = \lambda_0$ if and only if the curves*

$$0 = \frac{1}{\eta}\mathrm{Im}\{x_\alpha(a+\mathrm{i}\eta)\}, \qquad \alpha = 1, 2$$

intersect at $(\lambda_0, 0)$ in the (a, η) plane.

Proof. For small η,

$$\begin{aligned}\frac{1}{\eta}\mathrm{Im}\{x(a+\mathrm{i}\eta)\} &= \frac{1}{2\mathrm{i}\eta}(x(a+\mathrm{i}\eta) - x(a-\mathrm{i}\eta)) \\ &= \frac{\mathrm{d}x}{\mathrm{d}\lambda}(a) + \mathrm{O}(\eta^2),\end{aligned}$$

from Taylor's theorem. The result then follows by setting $a = \lambda_0$ and $\eta = 0$. □

We now consider a family of parametrized curves $x_t(\lambda) = \left(x_t^1(\lambda), x_t^2(\lambda)\right)$. As t varies, the geometry of the curve can change with swallowtails forming and disappearing.

Proposition 4.2. *If a swallowtail on the curve $x_t(\lambda)$ collapses to a point where $\lambda = \lambda_0$ when $t = t_0$, then*

$$\frac{\mathrm{d}x_{t_0}}{\mathrm{d}\lambda}(\lambda_0) = \frac{\mathrm{d}^2x_{t_0}}{\mathrm{d}\lambda^2}(\lambda_0) = 0.$$

Proof. This is a simple consequence of the two generalized cusps in a swallowtail coinciding. □

Similarly, we can consider the effect of a complex double point joining the curve.

Proposition 4.3. *If a complex double point joins the curve $x_t(\lambda)$ at $\lambda = \lambda_0$ when $t = t_0$, then*

$$\frac{\mathrm{d}x_{t_0}}{\mathrm{d}\lambda}(\lambda_0) = \frac{\mathrm{d}^2x_{t_0}}{\mathrm{d}\lambda^2}(\lambda_0) = 0.$$

Proof. If a complex double point has joined the curve then for $\alpha = 1, 2$, the curves

$$\frac{1}{\eta}\mathrm{Im}\left\{x_{t_0}^\alpha(a + \mathrm{i}\eta)\right\} = 0$$

must both intersect at $(\lambda_0, 0)$ and so the surfaces

$$z_\alpha(a, \eta) = \frac{1}{\eta}\mathrm{Im}\left\{x_{t_0}^\alpha(a + \mathrm{i}\eta)\right\}$$

must have critical points at $(\lambda_0, 0)$ where $z_\alpha = 0$.

Therefore, since

$$z_\alpha(a, \eta) = \frac{\mathrm{d}x_t^\alpha}{\mathrm{d}\lambda}(a) + \mathrm{O}(\eta^2),$$

we know that we must satisfy

$$\frac{\mathrm{d}x_{t_0}^\alpha}{\mathrm{d}\lambda}(\lambda_0) = \frac{\mathrm{d}^2x_{t_0}^\alpha}{\mathrm{d}\lambda^2}(\lambda_0) = 0, \qquad \alpha = 1, 2,$$

giving the result. □

Therefore, we have a necessary condition for the formation or destruction of a swallowtail and for complex double points to join or leave the main curve. This leads us to the following definition.

Definition 4.4. *A family of parametrized curves $x_t(\lambda)$ (where λ is some intrinsic parameter) for which*

$$\frac{\mathrm{d}x_{t_0}}{\mathrm{d}\lambda}(\lambda_0) = \frac{\mathrm{d}^2x_{t_0}}{\mathrm{d}\lambda^2}(\lambda_0) = 0$$

is said to have a point of swallowtail perestroika when $\lambda = \lambda_0$ and $t = t_0$.

This is a necessary condition for the birth of a swallowtail. However, as with our definition of generalized cusps, we have again not ruled out further degeneracy at the point. Although such points satisfy the definition of a generalized cusp, provided there is no further degeneracy, the curve will not actually appear cusped. As Cayley highlighted, these points are barely distinguishable from an ordinary point of the curve [13].

Lemma 4.5. *If*

$$\frac{\mathrm{d}x_{t_0}}{\mathrm{d}\lambda}(\lambda_0) = \frac{\mathrm{d}^2x_{t_0}}{\mathrm{d}\lambda^2}(\lambda_0) = 0 \qquad \textit{and} \qquad \frac{\mathrm{d}^3x_{t_0}}{\mathrm{d}\lambda^3}(\lambda_0) \neq 0,$$

then there is a well-defined normal to the curve $x_{t_0}(\lambda)$ at $\lambda = \lambda_0$.

Proof. The normal to a curve is given by $\tilde{n} = \hat{\tau}'(\lambda)$ where $\hat{\tau}$ denotes the unit tangent vector to the curve. Assuming $x_t'(\lambda)$ is real analytic,

$$x_t'(\lambda) = ((\lambda - \lambda_0)^2 F(\lambda), (\lambda - \lambda_0)^2 G(\lambda)),$$

where $(F(\lambda_0), G(\lambda_0)) \neq 0$. The result then follows. □

4.1. The complex caustic

We now consider a parametrization of the caustic $x_t(\lambda) \in \mathbb{R}^2$ in which we assume that $x_t(\lambda) = \Phi_t\big(\lambda, x^2_{0,\mathrm{C}}(\lambda)\big)$ where $\lambda \mapsto \big(\lambda, x^2_{0,\mathrm{C}}(\lambda)\big)$ for $\lambda \in \mathbb{R}$ denotes the pre-caustic parametrization, $x^2_{0,\mathrm{C}}(\lambda) \in \mathbb{R}$. Until now we have restricted the parameter to real values only, and have thus been ignoring isolated points which correspond to complex parameter values. We shall refer to the full caustic, in which we allow the parameter to vary over the complex plane, as the *complex caustic*.

We know that if x is a point on the caustic, that is $x = x_t(\lambda)$, then

$$f'_{(x,t)}(\lambda) = f''_{(x,t)}(\lambda) = 0.$$

Therefore, by considering the complex caustic, we are determining solutions $a = a_t$ and $\eta = \eta_t$ to the equations

$$f'_{(x,t)}(a + \mathrm{i}\eta) = f''_{(x,t)}(a + \mathrm{i}\eta) = 0,$$

where $x \in \mathbb{R}^2$. Hence, we can call the complex double points of the caustic *complex critical inflexions of* f. Moreover, we are particularly interested in these points if they join the main caustic at some finite critical time t_c. That is, we want there to exist a finite value $t_c > 0$ such that $\eta_t \to 0$ as $t \uparrow t_c$. If this holds, then a swallowtail can develop at the critical time t_c. Clearly, if $\eta_t \to 0$ as $t \downarrow t_c$, then a swallowtail will disappear. This can be expressed in terms of derivatives of the reduced action function.

Theorem 4.6. *For a two-dimensional caustic, assume that $x_t(\lambda)$ is a real analytic function. If at a time t_c a swallowtail perestroika occurs on the caustic, then $x = x_t(\lambda)$ is a real solution for x to*

$$f'_{(x,t_c)}(\lambda) = f''_{(x,t_c)}(\lambda) = f'''_{(x,t_c)}(\lambda) = f^{(4)}_{(x,t_c)}(\lambda) = 0.$$

Proof. As before, we differentiate the equation $0 = f''_{(x_t(\lambda),t)}(\lambda)$ giving

$$0 = \frac{\mathrm{d}x_t}{\mathrm{d}\lambda} \cdot \nabla_x f''_{(x_t(\lambda),t)}(\lambda) + f'''_{(x_t(\lambda),t)}(\lambda).$$

Differentiating again gives

$$0 = \frac{\mathrm{d}x_t}{\mathrm{d}\lambda} \cdot \frac{\partial}{\partial\lambda}\{\nabla_x f''\} + \frac{\mathrm{d}^2 x_t}{\mathrm{d}\lambda^2} \cdot \nabla_x f'' + \frac{\mathrm{d}x_t}{\mathrm{d}\lambda} \cdot \nabla_x f''' + f^{(4)}_{(x_t(\lambda),t)}(\lambda),$$

and the result follows. □

As before, we have the converse.

Theorem 4.7. *For a two-dimensional caustic, assume that $x_t(\lambda)$ is a real analytic function. If at a time t_c there is a real solution for x to*

$$f'_{(x,t_c)}(\lambda) = f''_{(x,t_c)}(\lambda) = f'''_{(x,t_c)}(\lambda) = f^{(4)}_{(x,t_c)}(\lambda) = 0,$$

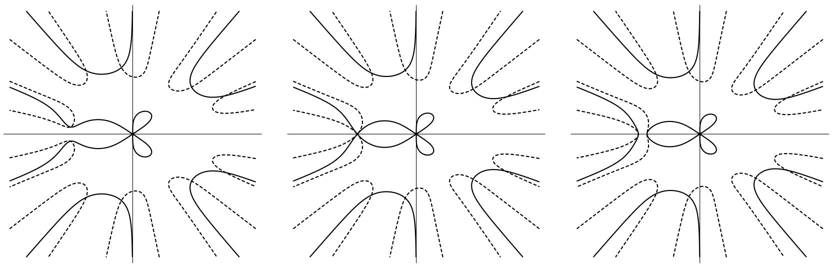

Figure 8. Curves $\mathrm{Im}\{x_t(a+\mathrm{i}\eta)\} = 0$ and $\mathrm{Im}\{y_t(a+\mathrm{i}\eta)\} = 0$ in (a, η) plane as we pass the critical time t_c.

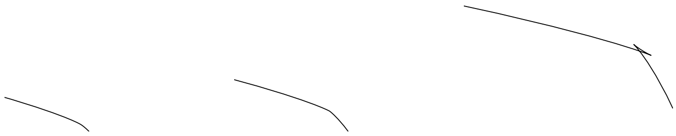

Figure 9. Caustic plotted at corresponding times.

and the vectors

$$\nabla_x f'_{(x,t_c)}(\lambda), \qquad \nabla_x f''_{(x,t_c)}(\lambda)$$

are linearly independent, then x is a point of swallowtail perestroika on the caustic.

Proof. Follows immediately from theorem 3.6 and the proof of theorem 4.6. □

Example 4.8. Let $V(x, y) = 0$, $k_t(x, y) \equiv 0$ and

$$S_0(x_0, y_0) = x_0^5 + x_0^6 y_0.$$

The equation of the caustic is $\lambda = a \in \mathbb{R}$

$$(x_t(a), y_t(a)) = \left(\frac{a}{5}(4 + 5a^3 t + 36a^{10}t^2), \frac{1}{30a^4 t}(-1 - 20a^3 t + 66a^{10}t^2)\right).$$

This has no cusps for times $t < t_c$ and two cusps for times $t > t_c$, where

$$t_c = \frac{4}{7}\sqrt{2}\left(\frac{33}{7}\right)^{\frac{3}{4}} = 2.5854\ldots.$$

This is an example of the above mechanism for forming a swallowtail as shown in figures 8 and 9. The dashed curves are $\mathrm{Im}\{x_t(a + \mathrm{i}\eta)\} = 0$ and the solid curves are $\mathrm{Im}\{y_t(a + \mathrm{i}\eta)\} = 0$. Conjugate pairs of intersections of these curves are the complex double points which give caustic swallowtails when $\eta \to 0$ and remain zero. In this example we have five complex double points before the critical time t_c and four afterwards. The remaining complex double points do not join the main caustic and so do not influence the behaviour of the caustic for real times.

4.2. Level surfaces

Unsurprisingly, these phenomena are not restricted to caustics. As one would expect, there is an interplay between the level surfaces and the caustics, characterized by their pre-images.

Let the level surface $\mathcal{A}(x_0, x, t) = c$ where $x, x_0 \in \mathbb{R}^2$ and $c \in \mathbb{R}$, be parametrized by $x_{(t,c)}(\lambda) = \Phi_t\big(\lambda, x^2_{0,\mathrm{ls}}(\lambda, c)\big)$, where $\lambda \mapsto \big(\lambda, x^2_{0,\mathrm{ls}}(\lambda, c)\big)$ is the pre-level surface parametrization with $x^2_{0,\mathrm{ls}}(\lambda, c) \in \mathbb{R}$. The geometric results of DTZ lead us to the following proposition.

Proposition 4.9. *Assume that in two dimensions at $x_0 \in \Phi_t^{-1} H_t^c \cap \Phi_t^{-1} C_t$ the normal to the pre-level surface $n(x_0) \neq 0$ and the normal to the pre-caustic $\tilde{n}(x_0) \neq 0$ so that neither the pre-level surface nor the pre-caustic are cusped at x_0. Then, $\tilde{n}(x_0)$ is parallel to $n(x_0)$ if and only if there is a generalized cusp on the caustic.*

Proof. Assume that the normal to the pre-caustic $\tilde{n}(x_0) \neq 0$ so that the pre-caustic is not cusped at x_0. Therefore, there is a cusp on the caustic at $\Phi_t(x_0)$ if and only if the tangent plane to the pre-caustic $\tilde{T}_{x_0}$ is spanned by the zero eigenvector e_0. However, the tangent plane to the pre-level surface T_{x_0} is spanned by e_0 when the pre-level surface intersects the pre-caustic. Thus, there is a cusp on the caustic if and only if the pre-caustic touches the pre-level surface. □

Corollary 4.10. *Assume that in two dimensions at $x_0 \in \Phi_t^{-1} H_t^c \cap \Phi_t^{-1} C_t$ the normal to the pre-level surface $n(x_0) \neq 0$ and the normal to the pre-caustic $\tilde{n}(x_0) \neq 0$ so that neither the pre-level surface nor the pre-caustic are cusped at x_0. Then, at $\Phi_t(x_0)$ there is a point of swallowtail perestroika on the level surface H_t^c if and only if there is a generalized cusp on the caustic C_t at $\Phi_t(x_0)$.*

Proof. When the pre-curves touch, there is a double point of contact between the pre-caustic and pre-level surface. However, there is a generalized cusp on a level surface whenever the pre-level surface intersects the pre-caustic. The double point of contact gives two cusps on the level surface which must coincide to produce a point of swallowtail perestroika. □

This corollary implies that a swallowtail can only form on a level surface curve where the curve meets a cusp on the caustic at which the pre-image of the level surface is sufficiently well behaved. Consider, for instance, the generic cusp example and zero level surface. This level surface meets the caustic at a cusp but the level surface does not have a triple point. Instead, it has a cusp because the pre-image consists of a parabola and line pair so that the normal is not well defined (see figure 3).

Thus, one would expect that the first two derivatives of the level surface being zero, would also force the first three derivatives of the reduced action function to be zero.

Firstly, it is clear that $f_{(x_{(t,c)}(\lambda),t)}(\lambda) = c$ and $f'_{(x_{(t,c)}(\lambda),t)}(\lambda) = 0$. Moreover, when the pre-level surface meets the pre-caustic, we have a root λ for $f''_{(x_{(t,c)}(\lambda),t)}(\lambda) = 0$. Now set

$$c_0 = f_{(x_t(\lambda),t)}(\lambda)|_{\lambda=\lambda_0},$$

where the caustic $x_t(\lambda)$ has a cusp when $\lambda = \lambda_0$. Assume as $c \uparrow c_0$ we have two roots of

$$f''_{(x_{(t,c)}(\lambda),t)}(\lambda) = 0,$$

λ_1 and λ_2, corresponding to cusps on the level surface, both tending to λ_0, so that the pre-surfaces will touch when $c = c_0$. Then, we will have a repeated root of this when $c = c_0$ and so

$$f'''_{(x_{(t,c_0)}(\lambda),t)}(\lambda) = 0.$$

Moreover, when $c > c_0$ these two roots will become complex conjugate pairs of points at which the complex caustic meets the level surface as expected by Klein's argument.

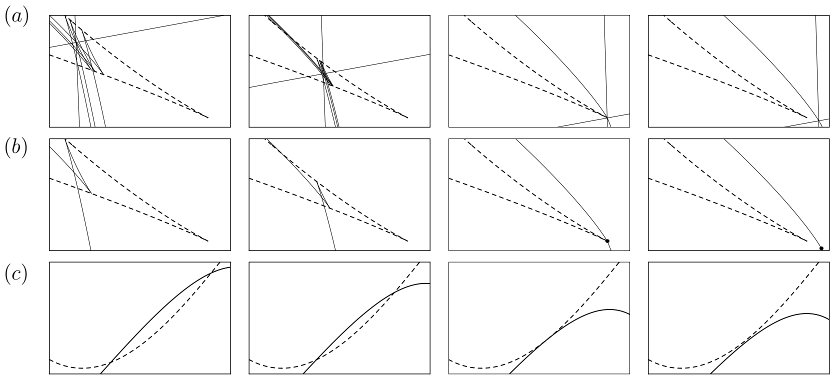

Figure 10. (*a*) All level surfaces (solid line) through a point as it crosses the caustic (dashed line) at a cusp, (*b*) one of these level surfaces with its complex double point, and (*c*) its real pre-image.

Example 4.11. Let $V(x, y) = 0$, $k_t(x, y) = 0$ and

$$S_0(x_0, y_0) = x_0^5 + x_0^6 y_0.$$

We consider the behaviour of the level surfaces through a given point at a fixed time as we move that point through a cusp on the caustic. This is illustrated in figure 10. Part (a) shows all of the level surfaces through a point demonstrating how three swallowtail level surfaces collapse together at the cusp to form a single level surface with a point of swallowtail perestroika. Parts (b) and (c) show how one of these swallowtails collapses on its own and how its pre-images behave.

4.3. Maxwell sets

When we cross the cool caustic, our minimal entropy solution for the Burgers equation jumps because the minimizing pre-image point vanishes. However, it is possible for the minimizer to jump without any pre-images disappearing. A jump will occur if we cross a point at which there are two different minimizers returning the same value of the action. It is this notion that leads us to the concept of Maxwell sets. We begin by returning to work in d dimensions.

Definition 4.12. *The Maxwell set is the set of all points $x \in \mathbb{R}^d$ such that if $x = \Phi_t(x_0)$ and $x = \Phi_t(\check{x}_0)$ where $x_0 \neq \check{x}_0$ and $x_0, \check{x}_0 \in \mathbb{R}^d$, then $\mathcal{A}(x_0, x, t) = \mathcal{A}(\check{x}_0, x, t)$.*

As with caustics, we define the cool part of the Maxwell set to be that part corresponding to the actual minimizer, and so it is the cool part of the Maxwell set across which our solution should be discontinuous. In terms of the reduced action function, the Maxwell set corresponds to values of x for which f has two critical points at the same height. If this occurs at the minimizing value then our solution should jump as shown in figure 11.

We now consider the two-dimensional case.

Lemma 4.13. *A point x is in the Maxwell set if and only if there is a Hamilton–Jacobi level surface with a point of self-intersection (crunode) at x.*

Proof. A consequence of the definition of the Maxwell set. □

Motivated by this lemma we have the following definition.

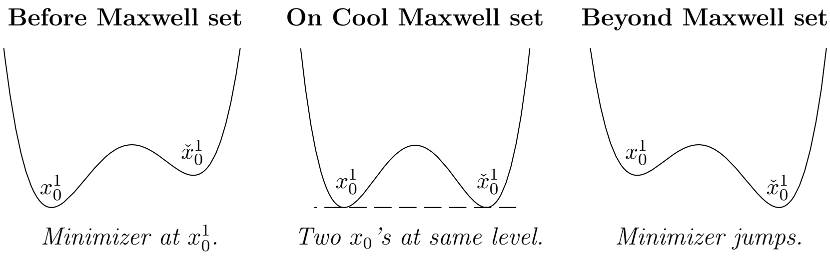

Figure 11. The graph of the reduced action function as x crosses the Maxwell set.

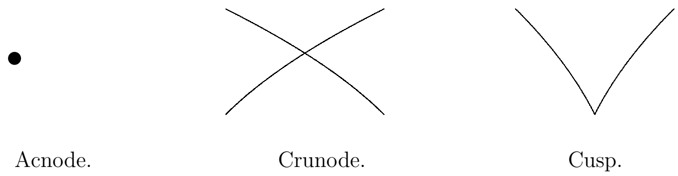

Figure 12. The classification of double points.

Definition 4.14. *The Maxwell–Klein set is the set of points which are non-cusp double points of some Hamilton–Jacobi level surface curve.*

(That is, these points are either complex double points or points of self-intersection of some Hamilton–Jacobi level surface.)

The results of section 2 tell us that, in the polynomial case, this set is simple to find. Here we use the fact that the cusp points of level surfaces sweep out the caustic. Thus, the equation of double points on the level surfaces must factorize into a product of factors corresponding to the caustic equation and the Maxwell–Klein equation. It is then only necessary to perform an analysis on the multiplicity of pre-images to extract the Maxwell set.

Theorem 4.15. *In the polynomial case, let D_t be the set of double points of the Hamilton–Jacobi level surfaces, C_t the caustic set and B_t the Maxwell–Klein set. Then, from Cayley and Klein's classification of double points as crunodes, acnodes and cusps, by definition $D_t = C_t \cup B_t$ and the corresponding defining algebraic equations factorize $D_t = C_t^n B_t^m$, where m, n are positive integers.*

(The Maxwell set is easy to find from the equation $B_t = 0$.)

Proof. Recall the classification of double points as crunodes, acnodes and cusps due to Cayley and Klein (see figure 12). We already know that cusps of the Hamilton–Jacobi level surfaces always occur on the caustic provided the pre-level surface is non-singular. Thus, we wish to find the remaining double points, the Klein complex double points (acnodes) and the Maxwell crossover points (crunodes). □

Theorem 4.16. *Let $\rho_{(t,c)}(x)$ be the resultant*

$$\rho_{(t,c)}(x) = R(f_{(x,t)}(\cdot) - c, f'_{(x,t)}(\cdot)),$$

where $x = (x_1, x_2)$. Then, $x \in D_t$ if and only if for some c

$$\rho_{(t,c)}(x) = \frac{\partial \rho_{(t,c)}}{\partial x_1}(x) = \frac{\partial \rho_{(t,c)}}{\partial x_2}(x) = 0.$$

Further,

$$D_t(x) = \gcd\left(\rho_t^1(x), \rho_t^2(x)\right),$$

where $\gcd(\cdot,\cdot)$ *denotes the greatest common divisor and* ρ_t^1 *and* ρ_t^2 *are the resultants*

$$\rho_t^1(x) = R\left(\rho_{(t,\cdot)}(x), \frac{\partial\rho_{(t,\cdot)}}{\partial x_1}(x)\right) \qquad \textit{and} \qquad \rho_t^2(x) = R\left(\frac{\partial\rho_{(t,\cdot)}}{\partial x_1}(x), \frac{\partial\rho_{(t,\cdot)}}{\partial x_2}(x)\right).$$

Proof. Recall that the equation of the level surface of Hamilton–Jacobi functions is merely the result of eliminating x_0^1 between the equations

$$f_{(x,t)}\left(x_0^1\right) = c \qquad \text{and} \qquad f'_{(x,t)}\left(x_0^1\right) = 0.$$

We form the resultant $\rho_{(t,c)}(x)$ using Sylvester's formula or the algorithm of Pohst and Zassenhaus [15]. The double points of the level surface must satisfy for some $c \in \mathbb{R}$:

$$\rho_{(t,c)}(x) = 0, \qquad \frac{\partial\rho_{(t,c)}}{\partial x_1}(x) = 0 \qquad \text{and} \qquad \frac{\partial\rho_{(t,c)}}{\partial x_2}(x) = 0.$$

Sylvester's formula proves all three equations are polynomial in c. To proceed we eliminate c between pairs of these equations using resultants, giving

$$R\left(\rho_{(t,\cdot)}(x), \frac{\partial\rho_{(t,\cdot)}}{\partial x_1}(x)\right) = \rho_t^1(x) \qquad \text{and} \qquad R\left(\frac{\partial\rho_{(t,\cdot)}}{\partial x_1}(x), \frac{\partial\rho_{(t,\cdot)}}{\partial x_2}(x)\right) = \rho_t^2(x).$$

Let $D_t = \gcd\left(\rho_t^1, \rho_t^2\right)$ be the greatest common divisor of the algebraic ρ_t^1 and ρ_t^2 which can be found using Euclid's algorithm. Then, $D_t(x) = 0$ is the equation of double points. □

Evidently, it is now a simple matter to factorize the expression $D_t = C_t^n B_t^m$, since C_t is known explicitly. The Maxwell–Klein set of double points is characterized by $B_t = 0$, so we only need to remove the Klein double points.

Clearly, if $f_{(x,t)}\left(x_0^1\right)$ is continuous in x_0^1, a Maxwell set can only exist in a region with three or more pre-images. Typically, the formation of a swallowtail on the caustic gives rise to a region inside the swallowtail with four pre-images and so will necessitate the formation of a Maxwell set, also typically of a swallowtail shape. Moreover, as has been explained, the swallowtail on the caustic leads to the formation of swallowtail level surfaces which are clearly connected to the Maxwell set by lemma 4.13.

The technique for finding hot/cool caustic boundaries outlined in section 3.3 corresponds to finding the intersections between the caustic and Maxwell set (as well as the points of self-intersection of the caustic). Further, the points we are interested in are the intersections with the cool Maxwell set. As a result, these points will be marked in the level surfaces as points at which there is a triple point consisting of a cusp meeting a line as shown in figure 7 part 2 and below in figure 13.

Example 4.17 (the generic cusp)**.** Let $V(x, y) = 0$, $k_t(x, y) = 0$ and $S_0(x_0, y_0) = x_0^2 y_0/2$. Here the equation of double points is

$$x^2(8 - 27t^2x^2 + 24ty + 24t^2y^2 + 8t^3y^3)^2 = 0,$$

where the second factor corresponds to the caustic. Thus, the Maxwell–Klein set is given by $x^2 = 0$, and a simple analysis of the multiplicity of the pre-image points tells us that the Maxwell set is given by $x = 0$ where $y > \frac{-1}{t}$. However, in this case, although the whole

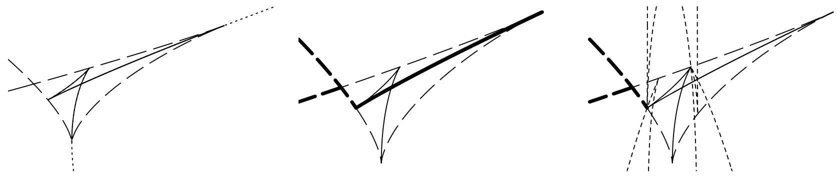

Figure 13. (*a*) The swallowtail caustic (long-dashed line) and Maxwell set (solid line) with the curve of Klein points (dotted line), (*b*) the cool parts of the caustic and Maxwell set highlighted in bold, and (*c*) the caustic and Maxwell set with the level surface through the caustic hot/cool boundary (short-dashed line).

Maxwell set is cool by definition, there will be no jump as we cross the Maxwell set because of the symmetry of the solution about the line $x = 0$.

Example 4.18 (the polynomial swallowtail)**.** Let $V(x, y) = 0$, $k_t(x, y) = 0$ and

$$S_0(x_0, y_0) = x_0^5 + x_0^2 y_0.$$

Again, the Maxwell–Klein set can be found by factorization as above giving

$$\begin{aligned}0 = {} & -675 + 52t^4 - t^8 + 3120t^3x - 224t^7x + 4t^{11}x - 38\,400t^2x^2 + 1408t^6x^2 + 128\,000tx^3\\ & - 5400ty + 312t^5y - 4t^9y + 12\,480t^4xy - 448t^8xy - 76\,800t^3x^2y\\ & -16\,200t^2y^2 + 624t^6y^2 - 4t^{10}y^2 + 12\,480t^5xy^2 - 21\,600t^3y^3\\ & +416t^7y^3 - 10\,800t^4y^4.\end{aligned}$$

The hot and cool parts of this caustic have already been calculated. The cool parts of the caustic and Maxwell set are shown in figure 13 by a thick line.

5. Some applications in two dimensions

5.1. Real turbulence and the ζ process

Definition 5.1. *The turbulent times t are times when the pre-level surface of the minimizing Hamilton–Jacobi function touches the pre-caustic. Such times t are zeros of a stochastic process $\zeta^c(\cdot)$, i.e. $\zeta^c(t) = 0$.*

These turbulent times are clearly times at which the number of cusps on the corresponding level surface will change. We begin with some minor generalizations of results in RTW and also [16].

Proposition 5.2. *Assume Φ_t is globally reducible with associated parametrization of the pre-caustic $\lambda \mapsto \big(\lambda, x_{0,\mathrm{C}}^2(\lambda)\big)$ in two dimensions. Then, the turbulence process at λ is given by*

$$\zeta^c(t) = f_{(x_t(\lambda_0),t)}(\lambda_0) - c,$$

where $f_{(x,t)}\big(x_0^1\big)$ is the reduced action function evaluated at points $x = x_t(\lambda_0) = \Phi_t\big(\lambda_0, x_0^2(\lambda_0)\big) \in C_t$, $\lambda = \lambda_0$ satisfying

$$\dot{X}_t(\lambda) \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\lambda}(\lambda) = 0,$$

where $\dot{X}_t(\lambda) = \dot{\Phi}_t\big(\lambda, x_{0,\mathrm{C}}^2(\lambda)\big)$ and $x_t(\lambda_0) \in C_t^c$, the cool part of the caustic.

Hence, there are three kinds of real stochastic turbulence:

(i) *cusped*, where there is a cusp on the caustic;
(ii) *zero speed*, where the Burgers fluid velocity is zero;
(iii) *orthogonal*, where the Burgers fluid velocity is orthogonal to the caustic.

Proof. The number of cusps on the relevant pre-level surface is

$$n_c(t) = \#\big\{\lambda \in \mathbb{R} : f_{(x_t(\lambda),t)}(\lambda) = c\big\},$$

where the roots $\lambda = \lambda_0$ correspond to points in the cool part of the caustic. The pre-surfaces touch when $n_c(t)$ changes, which occurs when

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{(x_t(\lambda),t)}(\lambda) = 0.$$

□

For stochastic turbulence to be intermittent, we require that the process $\zeta^c(t)$ is recurrent.

Proposition 5.3. *Let $V(x, y) = 0$, $k_t(x, y) = x$ and*

$$S_0(x_0, y_0) = f(x_0) + g(x_0)y_0,$$

where f, g, f' and g' are zero at $x_0 = a$ but $g''(a) \neq 0$. Then, for orthogonal turbulence at a, the zeta process is

$$\zeta^c(t) = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s\, \mathrm{d}s - \frac{\epsilon^2}{2}\int_0^t W_s^2\, \mathrm{d}s - c.$$

We note the following result of RTW [10] (also see for results on periodic systems).

Lemma 5.4. *Let W_t be a $BM(\mathbb{R})$ process starting at 0, c any real constant and*

$$Y_t = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s\, \mathrm{d}s - \frac{\epsilon^2}{2}\int_0^t W_s^2\, \mathrm{d}s - c.$$

Then, with probability 1, there exists a sequence of times $t_n \nearrow \infty$ such that

$$Y_{t_n} = 0 \qquad \text{for every } n.$$

Here we recapitulate our belief that cusped turbulence will be the most important. As we have shown, when the cusp on the caustic passes through a level surface, it forces a swallowtail to form on the level surface. The points of self-intersection of this swallowtail form the Maxwell set.

We can also use a lemma of Kac [17] to provide simple integral expressions for the number of cusps on the level surfaces and for the number of cusps on the caustic.

Lemma 5.5 (Kac's lemma)**.** *If $f(x)$ is continuous for $a \leqslant x \leqslant b$ and continuously differentiable for $a < x < b$ then, assuming $f(x)$ has a finite number of turning points, the number of zeros of $f(x)$ in (a, b) is given by*

$$n(a, b; f) = \lim_{R\to\infty} (2\pi)^{-1} \int_{-R}^{R}\int_a^b \cos(\xi f(x))|f'(x)|\, \mathrm{d}x\, \mathrm{d}\xi,$$

where multiple zeros are counted once and if either a or b is a zero it is counted as $\frac{1}{2}$.

Theorem 5.6. *Let $\lambda \mapsto x_t(\lambda)$ be a parametrization of the caustic derived from the pre-caustic x_0^1 parametrization. The number of generalized cusps on a level surface H_t^c is given by*

$$\lim_{R\to\infty} (2\pi)^{-1} \int_{-R}^{R}\int_a^b \cos\left(\xi\left\{f_{(x_t(\lambda),t)}(\lambda) - c\right\}\right) \left|\frac{\mathrm{d}x_t}{\mathrm{d}\lambda}(\lambda) \cdot \nabla_x f_{(x_t(\lambda),t)}(\lambda)\right| \mathrm{d}\lambda\, \mathrm{d}\xi.$$

Moreover, the number of generalized cusps on the caustic C_t is given by

$$\lim_{R\to\infty}(2\pi)^{-1}\int_{-R}^{R}\int_a^b \cos\left(\xi\left\{f'''_{(x_t(\lambda),t)}(\lambda)\right\}\right)\left|\frac{\mathrm{d}x_t}{\mathrm{d}\lambda}(\lambda)\cdot\nabla_x f'''_{(x_t(\lambda),t)}(\lambda)+f^{(4)}_{(x_t(\lambda),t)}(\lambda)\right|\mathrm{d}\lambda\,\mathrm{d}\xi,$$

if the vectors

$$\nabla_x f'_{(x_t(\lambda),t)}(\lambda),\qquad \nabla_x f''_{(x_t(\lambda),t)}(\lambda)$$

are linearly independent for all $\lambda\in\mathbb{R}$, with $f_{(x_t(\lambda),t)}(\lambda)=c$ or $f'''_{(x_t(\lambda),t)}(\lambda)=0$.

Proof. A simple consequence of Kac's lemma 5.5, the geometric results of DTZ and theorem 3.6. □

Observe that formally this is a consequence of the result that, if at time t there are n_t cusps on the caustic at $\lambda_1^c(t),\lambda_2^c(t),\ldots,\lambda_{n_t}^c(t)$, then

$$\sum_{i=1}^{n_t}\delta\left\{\lambda-\lambda_i^c(t)\right\}\mathrm{d}\lambda=\delta\left\{f'''_{(x_t(\lambda),t)}(\lambda)\right\}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f'''_{(x_t(\lambda),t)}(\lambda)\right)\right|\mathrm{d}\lambda$$

and

$$\delta\left\{f'''_{(x_t(\lambda),t)}(\lambda)\right\}=(2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\exp\left\{\mathrm{i}\xi f'''_{(x_t(\lambda),t)}(\lambda)\right\}.$$

5.2. *Complex turbulence and the resultant η process*

Let $\left(\lambda,x_{0,\mathrm{C}}^2(\lambda)\right)$ denote the pre-caustic at time t so that $x_t(\lambda)=\Phi_t\left(\lambda,x_{0,\mathrm{C}}^2(\lambda)\right)$ is a parametrization of the caustic. When $\mathrm{Im}\left\{\Phi_t\left(a+\mathrm{i}\eta,x_{0,\mathrm{C}}^2(a+\mathrm{i}\eta)\right)\right\}$ is random, the values of $\eta(t)$ for which this imaginary part is zero will form a stochastic process. As we have established, these zeros correspond to solutions of

$$f'_{(x,t)}(\lambda)=f''_{(x,t)}(\lambda)=f'''_{(x,t)}(\lambda)=f^{(4)}_{(x,t)}(\lambda)=0.$$

To find these points we need the following lemma.

Lemma 5.7. *Let g and h be polynomials of degrees m and n, respectively, with no common roots or zeros. Let $f=gh$ be the product polynomial. Then, the resultant*

$$R(f,f')=(-1)^{mn}\left(\frac{(N-1)!}{f^{(N)}(0)}\,\frac{g^{(m)}(0)}{(m-1)!}\,\frac{h^{(n)}(0)}{(n-1)!}\right)^N R(g,g')R(h,h')R(g,h)^2,$$

where $N=m+n$ and $R(g,h)\neq 0$.

Proof. Recall that

$$R(f,f')=R(gh,gh'+hg')=\left(\frac{f^{(N)}(0)}{(N-1)!}\right)^{-N}\prod_{\substack{w\in Z(g)\\ w\in Z(h)}}(gh'+hg')(w),$$

where $Z(g)$ denotes the set of zeros of g and $Z(h)$ those of h. The result then follows by evaluating the product. □

Since $f'_{(x_t(\lambda),t)}\left(x_0^1\right)$ is a polynomial in x_0 with real coefficients, its zeros are real or occur in complex conjugate pairs. Of the real roots, $x_0=\lambda$ is repeated. So

$$f'_{(x_t(\lambda),t)}\left(x_0^1\right)=\left(x_0^1-\lambda\right)^2 Q_{(\lambda,t)}\left(x_0^1\right)H_{(\lambda,t)}\left(x_0^1\right),$$

where Q is the product of quadratic factors

$$Q_{(\lambda,t)}\left(x_0^1\right) = \prod_{i=1}^{q} \left\{\left(x_0^1 - a_t^i\right)^2 + \left(\eta_t^i\right)^2\right\}$$

and $H_{(\lambda,t)}\left(x_0^1\right)$ is the product of real factors corresponding to real zeros. This gives

$$f'''_{(x_t(\lambda),t)}\left(x_0^1\right)\big|_{x_0^1=\lambda} = 2\prod_{i=1}^{q} \left\{\left(\lambda - a_t^i\right)^2 + \left(\eta_t^i\right)^2\right\} H_{(\lambda,t)}(\lambda).$$

We now assume that the real roots of H are distinct as are the complex roots of Q. Denoting $f'''_{(x_t(\lambda),t)}\left(x_0^1\right)\Big|_{x_0^1=\lambda}$ by $f_t'''(\lambda)$, etc, a simple calculation gives

$$\begin{aligned}
&\left|R_\lambda\left(f_t'''(\lambda), f_t^{(4)}(\lambda)\right)\right| \\
&\quad = K_t \prod_{k=1}^{q} \left(\eta_t^k\right)^2 \prod_{j\neq k} \left\{\left(a_t^k - a_t^j\right)^4 + 2\left(\left(\eta_t^k\right)^2 + \left(\eta_t^j\right)^2\right)\left(a_t^k - a_t^j\right)^2 + \left(\left(\eta_t^k\right)^2 - \left(\eta_t^j\right)^2\right)^2\right\} \\
&\quad\quad \times |R_\lambda(H, H')||R_\lambda(Q, H)|^2,
\end{aligned}$$

K_t being a positive constant. Thus, the condition for a swallowtail perestroika to occur is that

$$\rho_\eta(t) := \left|R_\lambda\left(f_t'''(\lambda), f_t^{(4)}(\lambda)\right)\right| = 0,$$

where we call $\rho_\eta(t)$ the *resultant eta process*. When the zeros of $\rho_\eta(t)$ form a perfect set, swallowtails will spontaneously appear and disappear on the caustic infinitely rapidly. As they do so, the geometry of the cool part of the caustic will rapidly change as the λ shaped sections typical of a swallowtail caustic appear and disappear. Moreover, Maxwell sets will be created and destroyed with each swallowtail that forms and vanishes adding to the turbulent nature of the solution in these regions. We call this *complex turbulence* occurring at the turbulent times which are the zeros of the resultant eta process.

This is in fact a very special form of the real turbulence outlined previously. As we have demonstrated, the condition for a swallowtail to form on the caustic is that its first two parametric derivatives are zero. This clearly fits into the scheme for real cusped turbulence. Thus, the pre-curves should touch at these points in a particular manner.

Corollary 5.8. *Assume that the pre-level surface and pre-caustic touch at a point $\tilde{x}_0$ where the normal to the pre-level surface is nonzero and well defined. Assume that this point of contact is of second order. Then, a swallowtail perestroika occurs on the caustic at that point. Moreover, at such a point, the first three derivatives of the parametrization of the level surface will be zero.*

5.3. The non-generic swallowtail

We conclude our two-dimensional work with an extended example. The non-generic swallowtail,

$$S_0(x_0, y_0) = x_0^5 + |x_0|^{\frac{3}{2}} y_0,$$

where $V(x, y) = 0$ and $k_t(x, y) = 0$, was first investigated by DTZ. For this initial condition, there is only one complex double point which can easily be found numerically, shown in figure 14.

An additional swallowtail forms on the caustic when this point joins the curve. Over time these swallowtails intersect one another and the caustic develops into a five-pointed star.

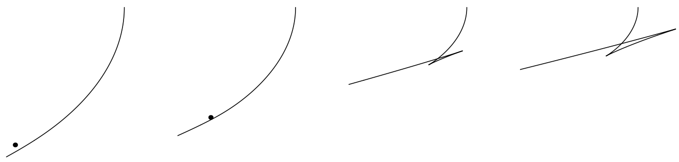

Figure 14. Complex double point joins the caustic.

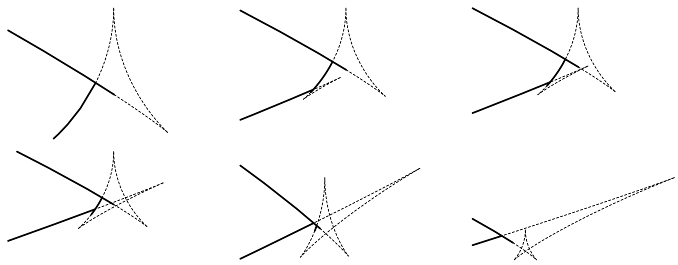

Figure 15. Hot and cool parts of the non-generic swallowtail before and after the swallowtail perestroika.

Figure 15 illustrates this development and also indicates the hot and cool parts as calculated numerically using our method. Cool parts are drawn in a thick solid line and hot parts with a dashed line.

6. Some applications in three or more dimensions

6.1. Hot and cool caustics in three dimensions

Using our new technique for identifying hot and cool parts, we can, for the first time, obtain an explicit expression for the hot and cool boundary on the three-dimensional swallowtail caustic. Previously, only a numerical approximation was possible due to the complexity of earlier techniques. Let

$$S_0(x_0, y_0, z_0) = x_0^7 + x_0^3 y_0 + x_0^2 z_0, \qquad V \equiv 0, \quad k_t \equiv 0.$$

The parametric form of the pre-caustic is

$$\begin{aligned} x_0 &= \lambda_1, \\ y_0 &= \lambda_2, \\ z_0(\lambda_1, \lambda_2) &= -\frac{1}{2t}\left(42t\lambda_1^5 - 9t^2\lambda_1^4 - 4t^2\lambda_1^2 + 6t\lambda_1\lambda_2 + 1\right). \end{aligned}$$

The caustic is

$$\begin{aligned} x_t(\lambda_1, \lambda_2) &= \lambda_1^2 t\left(-35\lambda_1^4 + 9t\lambda_1^3 + 4t\lambda_1 - 3\lambda_2\right), \\ y_t(\lambda_1, \lambda_2) &= \lambda_2 + t\lambda_1^3, \\ z_t(\lambda_1, \lambda_2) &= -21\lambda_1^5 + \frac{9}{2}t\lambda_1^4 + 3t\lambda_1^2 - 3\lambda_1\lambda_2 - \frac{1}{2t}. \end{aligned}$$

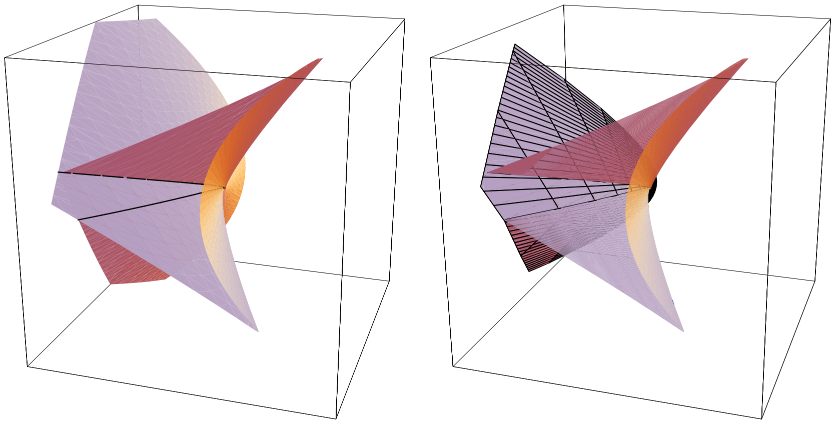

Figure 16. The 3D swallowtail caustic.

Then, following the earlier definitions, the boundary is given by finding the repeated roots of

$$\begin{aligned}\tilde{F}(x_0) &= -30\lambda_1^4 - 2\lambda_2 + 3\lambda_1 t + 8\lambda_1^3 t - 20\lambda_1^3 x_0 + t x_0 + 6\lambda_1^2 t x_0 - 12\lambda_1^2 x_0^2 + 3\lambda_1 t x_0^2 \\ &\quad - 6\lambda_1 x_0^3 + t x_0^3 - 2x_0^4, \\ \tilde{G}(x_0) &= -35\lambda_1^4 - 3\lambda_2 + 4\lambda_1 t + 9\lambda_1^3 t - 28\lambda_1^3 x_0 + 2t x_0 + 9\lambda_1^2 t x_0 - 21\lambda_1^2 x_0^2 + 6\lambda_1 t x_0^2 \\ &\quad - 14\lambda_1 x_0^3 + 3t x_0^3 - 7x_0^4.\end{aligned}$$

We can then find the boundary (on the pre-caustic) explicitly as

$$\begin{aligned}\lambda_2 &= \lambda_2(\lambda_1) \\ &= \begin{cases} \frac{1}{4096}\big(-35952\lambda_1^4 + 4608\lambda_1 t + 6176\lambda_1^3 t + 256t^2 + 408\lambda_1^2 t^2 + 72\lambda_1 t^3 \\ \quad + 9t^4 + \{P_1(\lambda_1) - Q_1(\lambda_1)\}^{\frac{1}{3}} + \{P_1(\lambda_1) + Q_1(\lambda_1)\}^{\frac{1}{3}}\big), \quad \lambda_1 < \lambda_c, \\ \frac{1}{87\,808}\big(-653\,072\lambda_1^4 + 87\,808\lambda_1 t + 98\,784\lambda^3 t + 6272t^2 + 7448\lambda_1^2 t^2 \\ \quad + 1512\lambda_1 t^3 + 243t^4 + \{P_2(\lambda_1) - Q_2(\lambda_1)\}^{\frac{1}{3}} + \{P_2(\lambda_1) + Q_2(\lambda_1)\}^{\frac{1}{3}}\big), \quad \lambda_1 > \lambda_c, \end{cases}\end{aligned}$$

where P_1 and P_2 are polynomials of degree 12 in λ_1 and t, and Q_1 and Q_2 are the square roots of polynomials of degree 24 in λ_1 and t. Also λ_c is the unique real solution for λ to the equation

$$70\lambda^3 - 15\lambda^2 t - t = 0.$$

It can then be easily demonstrated that the point on the caustic $x_t(\lambda_1, \lambda_2)$ is

(i) hot if $\lambda_2 < \lambda_2(\lambda_1)$;
(ii) cool if $\lambda_2 \geqslant \lambda_2(\lambda_1)$.

This is illustrated in figure 16 where the hatched areas represent the cool parts and the plain areas the hot parts.

6.2. The complex caustic in three dimensions

We consider how to extend our work on the complex caustic to a three-dimensional setting. There is no immediate analogue of Klein's work for three dimensions and so we consider instead what can be gained from the derivatives of the reduced action function. We have already found a geometrical interpretation for zeros of each of the first four derivatives in terms of the subcaustic, and so our attention turns to the fifth derivative. The natural way to extend our work is to reduce the three-dimensional case to a two-dimensional setting so that we can again apply Klein's ideas. We achieve this by considering the subcaustic and its projection onto each of the planes $x = 0$, $y = 0$ and $z = 0$. Setting the first three derivatives of f to zero forces us onto the subcaustic, and so we want to consider when complex double points of the caustic join the subcaustic. That is, we want to solve

$$\frac{1}{\eta}\,\mathrm{Im}\{\Phi_t(a+\mathrm{i}\eta, \lambda_2(a+\mathrm{i}\eta), z_t(a+\mathrm{i}\eta, \lambda_2(a+\mathrm{i}\eta)))\} = 0,$$

where $\lambda_2 = \lambda_2(\lambda_1)$ denotes the equation of the pre-subcaustic. However, this gives us three equations in the two unknowns a and η. Thus, we are forced to consider the projections of the subcaustic onto three orthogonal planes.

As in the two-dimensional case, we can consider when each of these projected curves has a point of swallowtail perestroika. If we find a time t_c and a parameter a at which a complex double point joins the projected subcaustic for each projection, then each will simultaneously have a point of swallowtail perestroika. Moreover, at such a time and position

$$\frac{\partial x_t^{\mathrm{sc}}}{\partial \lambda_1} = \frac{\partial^2 x_t^{\mathrm{sc}}}{\partial \lambda_1^2} = 0,$$

which leads us to the following proposition regarding the reduced action functional.

Proposition 6.1. *If each of the projected subcaustics has a point of swallowtail perestroika at a time t_c when $\lambda_1 = a$, then there is a real solution x to*

$$f'_{(x,t_c)}(a) = f''_{(x,t_c)}(a) = f'''_{(x,t_c)}(a) = f^{(4)}_{(x,t_c)}(a) = f^{(5)}_{(x,t_c)}(a) = 0.$$

Proof. Firstly, if $x = x_t^{\mathrm{sc}}(a)$, the subcaustic. Then, $0 = f'''_{(x_t^{\mathrm{sc}}(a),t)}(a)$ and differentiating with respect to a gives

$$0 = \frac{\mathrm{d}x_t^{\mathrm{sc}}}{\mathrm{d}a}\cdot\nabla_x f'''_{(x_t^{\mathrm{sc}}(a),t)}(a) + f^{(4)}_{(x_t^{\mathrm{sc}}(a),t)}(a).$$

Differentiating again gives

$$0 = \frac{\mathrm{d}x_t^{\mathrm{sc}}}{\mathrm{d}a}\cdot\frac{\partial}{\partial a}\{\nabla_x f'''\} + \frac{\mathrm{d}^2 x_t^{\mathrm{sc}}}{\mathrm{d}a^2}\cdot\nabla_x f''' + \frac{\mathrm{d}x_t^{\mathrm{sc}}}{\mathrm{d}a}\cdot\nabla_x f^{(4)} + f^{(5)}_{(x_t^{\mathrm{sc}}(a),t)}(a),$$

and the result follows. □

The formation of a swallowtail on each of the projections of the subcaustic produces an interesting pyramidical shape on the caustic.

Example 6.2. Let $V \equiv 0$, $k_t \equiv 0$ and

$$S_0(x_0, y_0, z_0) = x_0^4 + x_0^3 + x_0^2 + x_0^5 y_0 + x_0^2 z_0.$$

Then (to 5 d.p.) at time $t = 5.980\,56$ when $x = -0.199\,789$, $y = 1.629\,76$, $z = -1.340\,06$ and $x_0 = 0.230\,91$, the first five derivatives of the reduced action functional will be zero. Therefore, from the preceding propositions, we would expect swallowtails to form on each of the three projections of the subcaustic (figure 17). Moreover, if we consider the caustic, two-dimensional swallowtails form on slices taken parallel to any of the axes. In fact a pyramid like structure forms on the caustic (figure 18).

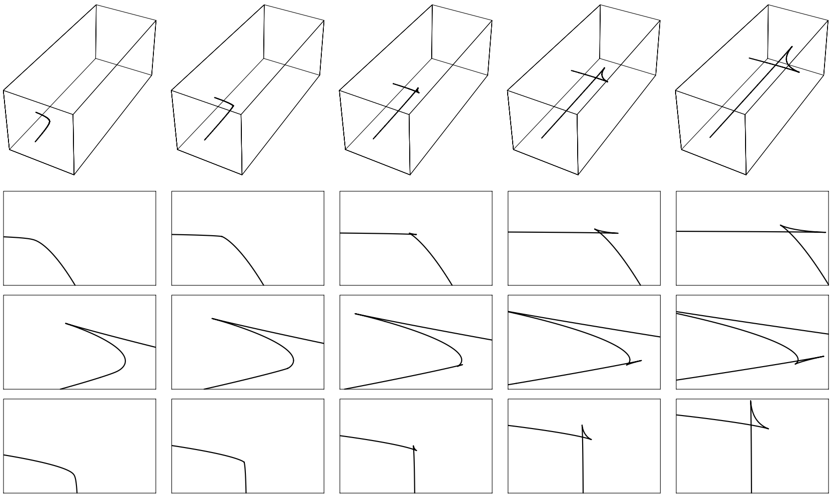

Figure 17. Subcaustic with projections at times $t = 5, 6, 7, 8, 9$.

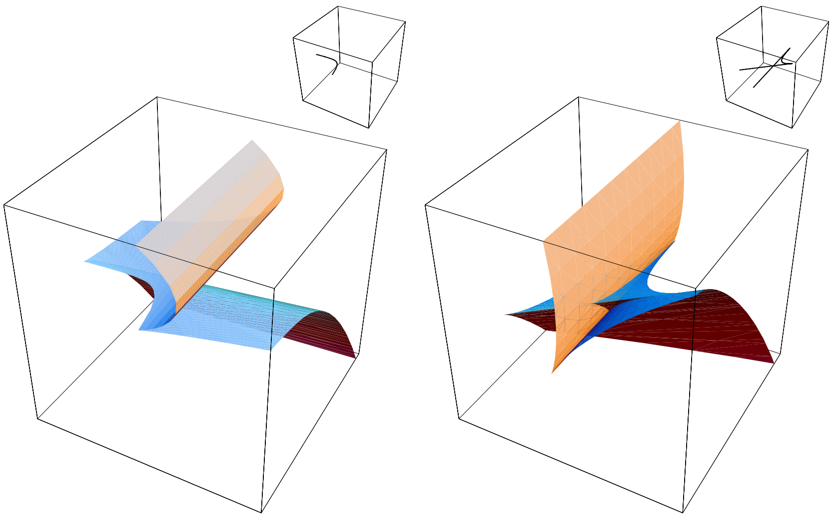

Figure 18. The caustic (with subcaustic inset) at times $t = 5$ and $t = 9$.

6.3. The Maxwell set in d dimensions

We now come to one of our main results. We present a simple extension of the geometric results of section 4.3, in particular to theorems 4.15 and 4.16. For polynomial $f_{(x,t)}(x_0^1)$, this enables us to state an explicit algebraic equation for the entire set of singularities for the inviscid limit of the Burgers fluid velocity in any dimension. Again from this, it is a simple matter to separate the caustic equation and then split the Maxwell set from the remaining Maxwell–Klein set. In this section we dispense with the need to refer to the detailed geometry of the level surfaces. We assume that the action $\mathcal{A}$ is globally reducible and polynomial in space and time variables.

Theorem 6.3. *If the associated reduced action function f is a polynomial, then the set of all possible discontinuities for a d-dimensional Burgers fluid velocity field in the inviscid limit is the double discriminant*

$$D(t) := D^c\{D^\lambda(f_{(x,t)}(\lambda) - c)\} = 0,$$

where $D^x(p(x))$ is the discriminant of the polynomial p with respect to x.

Proof. The discriminant of $(f_{(x,t)}(\lambda) - c)$ with respect to λ is simply the resultant

$$0 = R(c) := R_\lambda(f_{(x,t)}(\lambda) - c, f'_{(x,t)}(\lambda)),$$

where according to Sylvester's formula

$$R(c) = \begin{vmatrix} f^{(m)}(\tilde\lambda)/m! & f^{(m-1)}(\tilde\lambda)/(m-1)! & \cdots & f'(\tilde\lambda) & f(\tilde\lambda)-c & 0 & \cdots & 0 \\ 0 & f^{(m)}(\tilde\lambda)/m! & \cdots & \cdots & f'(\tilde\lambda) & f(\tilde\lambda)-c & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & f(\tilde\lambda)-c \\ f^{(m)}(\tilde\lambda)/m! & f^{(m-1)}(\tilde\lambda)/(m-1)! & \cdots & f'(\tilde\lambda) & 0 & 0 & \cdots & 0 \\ 0 & f^{(m)}(\tilde\lambda)/m! & \cdots & \cdots & f'(\tilde\lambda) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & f'(\tilde\lambda) \end{vmatrix},$$

where the resultant is independent of $\tilde\lambda$. Therefore, we can set $\tilde\lambda = x_0^1(i)(x,t)$ where $x_0^1(i)(x,t)$ is any one of the real or complex roots of $f'_{(x,t)}(\lambda) = 0$. Hence, $R(c)$ must have a factor $\left(f_{(x,t)}(x_0^1(i)(x,t)) - c\right)$ for each $x_0^1(i)(x,t)$ real or complex. Since $R(c)$ is a polynomial of degree m, the degree of λ in $f'_{(x,t)}(\lambda)$, we see that

$$R(c) = \prod_{i=1}^{m} \left(f_{(x,t)}\left(x_0^1(i)(x,t)\right) - c\right),$$

where $x_0^1(i)(x,t)$ is an enumeration of the real and complex roots λ of $f'_{(x,t)}(\lambda) = 0$.

Moreover, the discriminant is given in terms of the zeros of $R(c)$ as

$$D^c(R(c)) = b_0^{2m-2} \prod_{i<j} \left(f_{(x,t)}\left(x_0^1(i)(x,t)\right) - f_{(x,t)}\left(x_0^1(j)(x,t)\right)\right)^2,$$

where b_0 is the leading coefficient of $R(c)$. This is zero when

(i) $x_0^1(i)(x,t) = x_0^1(j)(x,t)$ and $i \neq j$, which corresponds to the complex caustic.
(ii) $x_0^1(i)(x,t) \neq x_0^1(j)(x,t)$ but $f_{(x,t)}\left(x_0^1(i)(x,t)\right) = f_{(x,t)}\left(x_0^1(j)(x,t)\right)$, which corresponds to the Maxwell–Klein set.

Hence, the equation $D(t) = 0$ contains the cool caustic and cool Maxwell set, and so contains all points of discontinuity of the minimal entropy solution of the Burgers equation. □

In the two-dimensional case, the curves defined by $D(t)$ above and D_t given in theorem 4.16 coincide. However, unlike the two-dimensional method, the powers of the corresponding factors can be found explicitly.

Lemma 6.4. *The equation of the caustic is equivalent to*

$$\prod_{i<j} \left(x_0^1(i)(x,t) - x_0^1(j)(x,t)\right)^2 = 0,$$

where we are taking the product over all real and complex roots x_0^1 of $f'_{(x,t)}\left(x_0^1\right) = 0$.

Proof. The caustic can be seen as the zeros of the discriminant of $f'_{(x,t)}\big(x_0^1\big)$ taken with respect to x_0^1. This discriminant is

$$\left(\frac{f^{(m)}(0)}{m!}\right)^{2m-2}\prod_{i<j}\big(x_0^1(i)(x,t)-x_0^1(j)(x,t)\big)^2,$$

giving the result. □

Lemma 6.5. *If f is a polynomial such that $f'(b)=f'(a)=0$, then*

$$f(b)-f(a)=(b-a)^3g(a,b),$$

for some polynomial g.

Proof. Assume, without loss of generality, that $a=0$. Then,

$$f(b)-f(0)=\int_0^b x(x-b)h(x)\,\mathrm{d}x,$$

where $f'(x)=x(x-b)h(x)$ for some polynomial $h(x)$. Differentiating with respect to b gives

$$f'(b)=-\int_0^b xh(x)\,\mathrm{d}x=\mathrm{O}(b^2),$$

and so the result follows. □

Theorem 6.6. *The double discriminant $D(t)$ factorizes as*

$$D(t)=b_0^{2m-2}(C_t)^3(B_t)^2$$

where $B_t=0$ is the equation of the Maxwell–Klein set and $C_t=0$ is the equation of the caustic. The expressions C_t and B_t defining the caustic and the Maxwell–Klein set are both algebraic in x and t.

Proof. From theorem 6.3 and lemma 6.4 we have that

$$\begin{aligned}D(t)&=b_0^{2m-2}\prod_{i<j}\big(f_{(x,t)}\big(x_0^1(i)(x,t)\big)-f_{(x,t)}\big(x_0^1(j)(x,t)\big)\big)^2\\&=b_0^{2m-2}\prod_{i<j}\big\{\big(x_0^1(i)(x,t)-x_0^1(j)(x,t)\big)^2\big\}^3\{p_{ij}(x,t)\}^2,\end{aligned}$$

where from lemma 6.5

$$f_{(x,t)}\big(x_0^1(i)(x,t)\big)-f_{(x,t)}\big(x_0^1(j)(x,t)\big)=\big(x_0^1(i)(x,t)-x_0^1(j)(x,t)\big)^3p_{ij}(x,t).$$

Moreover,

$$\prod_{i<j}p_{ij}(x,t),$$

is a symmetric function of the roots of $f'_{(x,t)}\big(x_0^1\big)=0$, and so, by the fundamental theorem of symmetric functions [11], is a polynomial in the coefficients of $f'_{(x,t)}\big(x_0^1\big)$ and so is algebraic in x and t. □

Example 6.7 (the butterfly)**.** Let $V(x,y,z)=0$, $k_t(x,y)=0$ and

$$S_0(x_0,y_0,z_0)=x_0^3y_0+x_0^2z_0.$$

The butterfly initial condition is the three-dimensional equivalent of the generic cusp. Evaluating the first discriminant gives $D^{\lambda}(f_{(x,t)}(\lambda) - c)$ as a polynomial of degree 5 in c. Therefore, the second discriminant can be found easily using the standard formula for the discriminant of the quintic.

Moreover, we can then perform the required factorization by simply dividing by the factor corresponding to the caustic to give the Maxwell–Klein equation

$$\begin{aligned}
&432(3x^2 - y^2) + 432(2xy + 25x^3y - 9xy^3 + 30x^2z - 12y^2z)t + 27(72x^2 + 500x^4 \\
&\quad + 3125x^6 - 24y^2 + 192x^2y^2 - 1125x^4y^2 - 36y^4 + 27x^2y^4 - 27y^6 \\
&\quad + 320xyz + 2400x^3yz - 1152xy^3z + 1920x^2z^2 - 960y^2z^2)t^2 \\
&\quad + 54(24xy + 510x^3y + 3750x^5y - 90xy^3 - 990x^3y^3 - 108x^3y^5 \\
&\quad + 288x^2z + 1000x^4z - 120y^2z + 576x^2y^2z - 1125x^4y^2z - 144y^4z \\
&\quad + 54x^2y^4z - 81y^6z + 640xyz^2 + 2400x^3yz^2 - 1728xy^3z^2 + 1920x^2z^3 \\
&\quad - 1280y^2z^3)t^3 + 9(129x^2 + 775x^4 - 43y^2 + 1536x^2y^2 + 21\,150x^4y^2 \\
&\quad - 159y^4 - 3807x^2y^4 - 81y^6 - 972x^2y^6 + 1152xyz + 12\,240x^3yz \\
&\quad - 3240xy^3z - 11\,880x^3y^3z + 5184x^2z^2 + 6000x^4z^2 - 2880y^2z^2 \\
&\quad + 6912x^2y^2z^2 - 2592y^4z^2 + 324x^2y^4z^2 - 972y^6z^2 + 7680xyz^3 + 9600x^3yz^3 \\
&\quad - 13\,824xy^3z^3 + 11\,520x^2z^4 - 11\,520y^2z^4)t^4 + 18(43xy + 673x^3y - 6xy^3 \\
&\quad + 4833x^3y^3 - 540xy^5 - 243xy^7 + 387x^2z + 775x^4z - 172y^2z + 3072x^2y^2z \\
&\quad - 477y^4z - 3807x^2y^4z - 162y^6z + 1728xyz^2 + 6120x^3yz^2 - 3240xy^3z^2 \\
&\quad + 3456x^2z^3 - 2880y^2z^3 + 2304x^2y^2z^3 - 1728y^4z^3 - 324y^6z^3 + 3840xyz^4 \\
&\quad - 3456xy^3z^4 + 2304x^2z^5 - 4608y^2z^5)t^5 + (345x^2 + 906x^4 - 115y^2 \\
&\quad + 6156x^2y^2 - 567y^4 + 19\,035x^2y^4 - 1215y^6 - 729y^8 + 4644xyz + 24\,228x^3yz \\
&\quad - 432xy^3z - 19\,440xy^5z + 13\,932x^2z^2 - 9288y^2z^2 + 55\,296x^2y^2z^2 \\
&\quad - 17\,172y^4z^2 - 2916y^6z^2 + 41\,472xyz^3 - 38\,880xy^3z^3 + 31\,104x^2z^4 \\
&\quad - 51\,840y^2z^4 - 15\,552y^4z^4 + 27\,648xyz^5 - 27\,648y^2z^6)t^6 + 2(115xy \\
&\quad + 743x^3y + 333xy^3 + 729xy^5 + 690x^2z - 345y^2z + 6156x^2y^2z - 1134y^4z \\
&\quad - 1215y^6z + 4644xyz^2 - 216xy^3z^2 + 4644x^2z^3 - 6192y^2z^3 - 5724y^4z^3 \\
&\quad + 10\,368xyz^4 - 10\,368y^2z^5)t^7 + (51x^2 - 17y^2 + 762x^2y^2 - 72y^4 - 54y^6 \\
&\quad + 920xyz + 1332xy^3z + 1380x^2z^2 - 1380y^2z^2 - 2268y^4z^2 + 6192xyz^3 \\
&\quad - 6192y^2z^4)t^8 + 2(17xy + 57xy^3 + 51x^2z - 34y^2z - 72y^4z + 460xyz^2 \\
&\quad - 460y^2z^3)t^9 + (3x^2 - y^2 - y^4 + 68xyz - 68y^2z^2)t^{10} + 2y(x - yz)t^{11} = 0.
\end{aligned}$$

Distinguishing the Maxwell set from the Maxwell–Klein set is a more complex matter in this example as there are five pre-images (real and complex) everywhere. We can rule out the existence of a Maxwell set below the caustic in figure 19 as there is only one real pre-image. However, above the caustic there are three real and two complex pre-images. We have included this to demonstrate how our simple method gives the algebraic equation of these very complicated surfaces. We have computed the Maxwell–Klein set in many more complex examples, including the three-dimensional polynomial swallowtail and the non-generic swallowtail, where the factorized expression runs to seven pages.

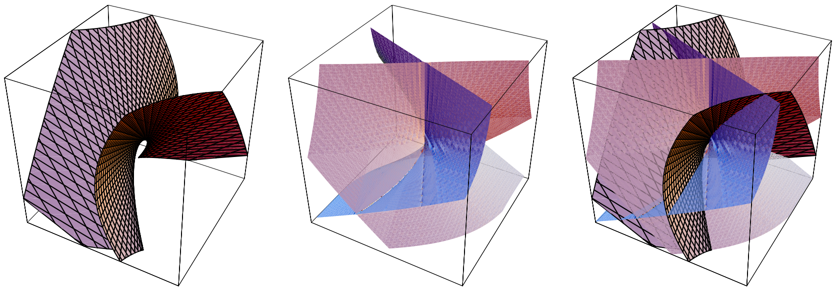

Figure 19. The caustic and Maxwell–Klein set at time $t = 1$.

7. Conclusion

We have seen how the reduced action function provides a powerful tool to simplify and analyse many aspects of the Burgers equation. The assumptions required for global reducibility appear to be quite restrictive. However, for local reducibility at x, we require only that in a neighbourhood of x there is at most one value of α such that

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha} = \frac{\partial^2 \mathcal{A}}{\left(\partial x_0^\alpha\right)^2} = 0 \qquad \text{when} \qquad \nabla_{x_0} \mathcal{A} = 0.$$

Therefore, to apply our results, we only have to avoid two second-order derivatives with respect to two different Cartesian coordinates being zero simultaneously. Thus, local reducibility is valid for a very general class of problems. With this property, we can easily establish the geometry and behaviour of the caustic and level surfaces. In particular, we can identify the points of turbulent behaviour and demonstrate the recurrent nature of the turbulence. Of the turbulent behaviours identified in two dimensions, cusped turbulence, which incorporates complex turbulence, appears to be the most important. Under this condition, not only can cusps appear and disappear on the level surfaces, swallowtails can form on both the caustics and level surfaces, changing the shape of the cool part of the caustic. Moreover, when a swallowtail forms on the caustic, a Maxwell set can be created adding further discontinuities to the Burgers velocity field. Finally, we have been able to state an explicit equation for the complete set of discontinuities for the inviscid limit of the Burgers fluid velocity in the d-dimensional polynomial case. This is a much simpler approach than the Clausius–Clapeyron equations traditionally used to identify the Maxwell set [18]. Needless to say, we have calculated numerous examples of Maxwell sets and the other phenomena presented in this paper but have only included a few because of restrictions on space.

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